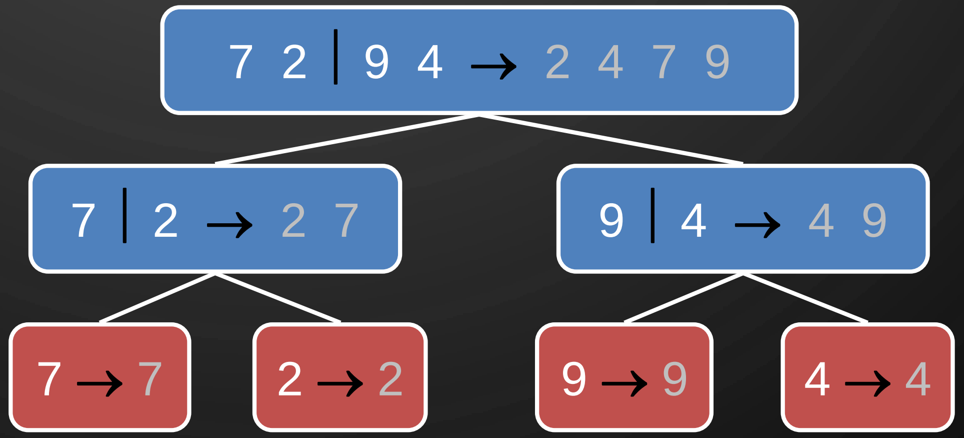
CHAPTER 11
SORTING, SETS, AND SELECTION

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN C++, GOODRICH, TAMASSIA AND MOUNT (WILEY 2004) AND SLIDES FROM NANCY M. AMATO

MERGE SORT



MERGE-SORT

- Merge-sort is based on the divide-and-conquer paradigm. It consists of three steps:
  - Divide: partition input sequence S into two sequences S_1 and S_2 of about $\frac{n}{2}$ elements each
  - Recur: recursively sort S_1 and S_2
  - Conquer: merge S_1 and S_2 into a sorted sequence

Algorithm $\text{mergeSort}(S, C)$

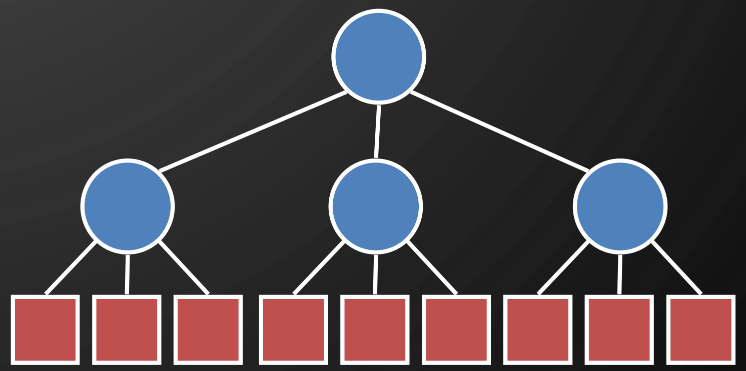
Input: Sequence S of n elements,
Comparator C

Output: Sequence S sorted according to C

1. **if** $S.size(\ ) > 1$
2. $\quad (S_1, S_2) \leftarrow \text{partition}\left(S, \frac{n}{2}\right)$
3. $\quad S_1 \leftarrow \text{mergeSort}(S_1, C)$
4. $\quad S_2 \leftarrow \text{mergeSort}(S_2, C)$
5. $\quad S \leftarrow \text{merge}(S_1, S_2)$
6. **return** S

DIVIDE AND CONQUER ALGORITHMS
ANALYSIS WITH RECURRENCE EQUATIONS

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data S into k (disjoint) subsets $S_1, S_2, \ldots, S_k$
  - Recur: solve the subproblems recursively
  - Conquer: combine the solutions for $S_1, S_2, \ldots, S_k$ into a solution for S
- The base case for the recursion are subproblems of constant size
- Analysis can be done using recurrence equations (relations)

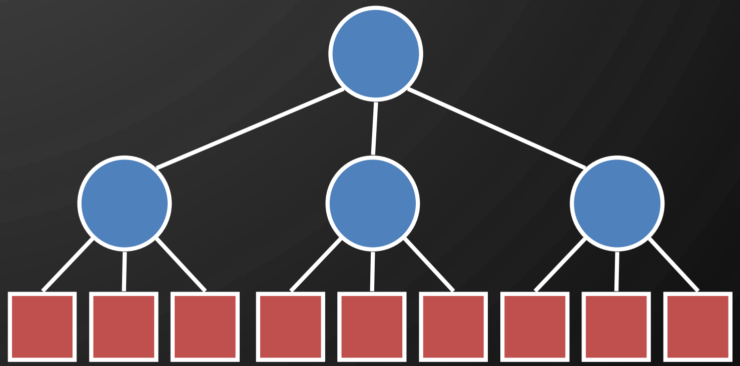


DIVIDE AND CONQUER ALGORITHMS ANALYSIS WITH RECURRENCE EQUATIONS

- When the size of all subproblems is the same (frequently the case) the recurrence equation representing the algorithm is:

$$T(n) = D(n) + kT\left(\frac{n}{c}\right) + C(n)$$

- Where
  - $D(n)$ is the cost of dividing S into the k subproblems $S_1, S_2, \ldots, S_k$
  - There are k subproblems, each of size $\frac{n}{c}$ that will be solved recursively
  - $C(n)$ is the cost of combining the subproblem solutions to get the solution for S



EXERCISE
RECURRENCE EQUATION SETUP

- Algorithm – transform multiplication of two n-bit integers I and J into multiplication of $\left(\frac{n}{2}\right)$-bit integers and some additions/shifts

1. Where does recursion happen in this algorithm?
2. Rewrite the step(s) of the algorithm to show this clearly.

Algorithm $\text{multiply}(I, J)$
Input: n-bit integers I, J
Output: $I * J$

```
if n > 1
    Split I and J into high and low order halves: I_h, I_l, J_h, J_l
    x_1 ← I_h * J_h; x_2 ← I_h * J_l; x_3 ← I_l * J_h; x_4 ← I_l * J_l
    Z ← x_1 * 2^n + x_2 * 2^(n/2) + x_3 * 2^(n/2) + x_4
else
    Z ← I * J
return Z
```

EXERCISE
RECURRENCE EQUATION SETUP

- Algorithm – transform multiplication of two n-bit integers I and J into multiplication of $\left(\frac{n}{2}\right)$-bit integers and some additions/shifts

3. Assuming that additions and shifts of n-bit numbers can be done in $O(n)$ time, describe a recurrence equation showing the running time of this multiplication algorithm

Algorithm $\text{multiply}(I, J)$
Input: n-bit integers I, J
Output: $I * J$

1. **if** $n > 1$
2. Split I and J into high and low order halves: I_h, I_l, J_h, J_l
3. $x_1 \leftarrow \text{multiply}(I_h, J_h)$; $x_2 \leftarrow \text{multiply}(I_h, J_l)$; $x_3 \leftarrow \text{multiply}(I_l, J_h)$; $x_4 \leftarrow \text{multiply}(I_l, J_l)$
4. $Z \leftarrow x_1 * 2^n + x_2 * 2^{\frac{n}{2}} + x_3 * 2^{\frac{n}{2}} + x_4$
5. **else**
6. $Z \leftarrow I * J$
7. **return** Z

EXERCISE
RECURRENCE EQUATION SETUP

- Algorithm – transform multiplication of two n-bit integers I and J into multiplication of $\left(\frac{n}{2}\right)$-bit integers and some additions/shifts
- The recurrence equation for this algorithm is:
  - $T(n) = 4T\left(\frac{n}{2}\right) + O(n)$
  - The solution is $O(n^2)$ which is the same as naïve algorithm

Algorithm $\text{multiply}(I, J)$
Input: n-bit integers I, J
Output: $I * J$

1. **if** $n > 1$
2. Split I and J into high and low order halves: I_h, I_l, J_h, J_l
3. $x_1 \leftarrow \text{multiply}(I_h, J_h)$; $x_2 \leftarrow \text{multiply}(I_h, J_l)$; $x_3 \leftarrow \text{multiply}(I_l, J_h)$; $x_4 \leftarrow \text{multiply}(I_l, J_l)$
4. $Z \leftarrow x_1 * 2^n + x_2 * 2^{\frac{n}{2}} + x_3 * 2^{\frac{n}{2}} + x_4$
5. **else**
6. $Z \leftarrow I * J$
7. **return** Z

NOW, BACK TO MERGESORT…

- The running time of Merge Sort can be expressed by the recurrence equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + M(n)$$

- We need to determine $M(n)$, the time to merge two sorted sequences each of size $\frac{n}{2}$.

Algorithm $\text{mergeSort}(S, C)$

Input: Sequence S of n elements,
Comparator C

Output: Sequence S sorted according to C

1. **if** $S.size(\;\;) > 1$
2. $(S_1, S_2) \leftarrow \text{partition}\left(S, \frac{n}{2}\right)$
3. $S_1 \leftarrow \text{mergeSort}(S_1, C)$
4. $S_2 \leftarrow \text{mergeSort}(S_2, C)$
5. $S \leftarrow \text{merge}(S_1, S_2)$
6. **return** S

MERGING TWO SORTED SEQUENCES

- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with $\frac{n}{2}$ elements and implemented by means of a doubly linked list, takes $O(n)$ time
  - $M(n) = O(n)$

Algorithm $merge(A, B)$

Input: Sequences A, B with $\frac{n}{2}$ elements each

Output: Sorted sequence of $A \cup B$

$S \leftarrow \emptyset$
while $\neg A.empty(\ ) \wedge \neg B.empty(\ )$
    if $A.front(\ ) < B.front(\ )$
        $S.insertBack(A.front(\ ));\ A.eraseFront(\ )$
    else
        $S.insertBack(B.front(\ ));\ B.eraseFront(\ )$
while $\neg A.empty(\ )$
    $S.insertBack(A.front(\ ));\ A.eraseFront(\ )$
while $\neg B.empty(\ )$
    $S.insertBack(B.front(\ ));\ B.eraseFront(\ )$
return S

AND THE COMPLEXITY OF MERGESORT…

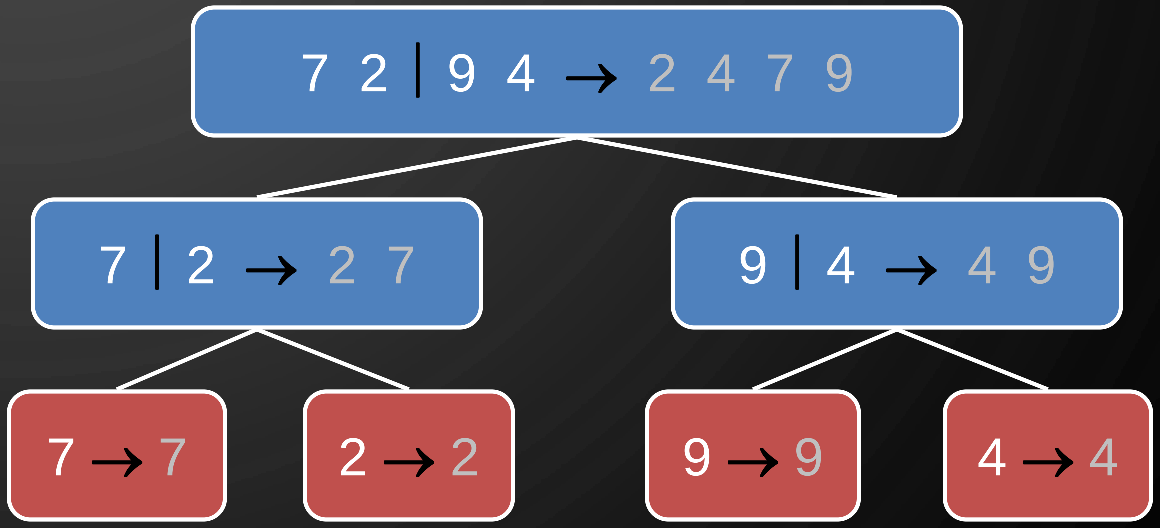
- So, the running time of Merge Sort can be expressed by the recurrence equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + M(n)$$
$$= 2T\left(\frac{n}{2}\right) + O(n)$$
$$= O(n \log n)$$

Algorithm mergeSort(S, C)
Input: Sequence S of n elements,
Comparator C
Output: Sequence S sorted according to C
1. **if** $S.size(\quad) > 1$
2. $(S_1, S_2) \leftarrow \text{partition}\left(S, \frac{n}{2}\right)$
3. $S_1 \leftarrow \text{mergeSort}(S_1, C)$
4. $S_2 \leftarrow \text{mergeSort}(S_2, C)$
5. $S \leftarrow \text{merge}(S_1, S_2)$
6. **return** S

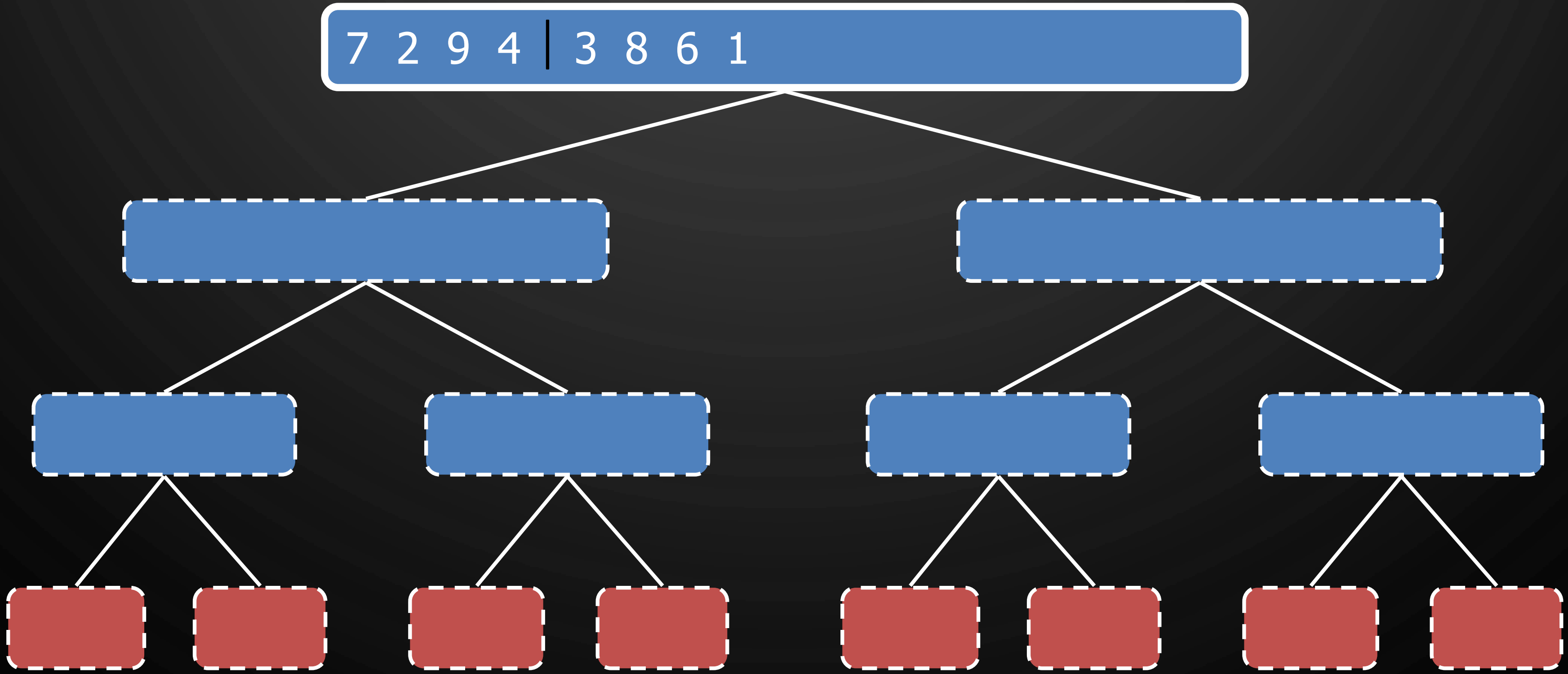
MERGE-SORT EXECUTION TREE (RECURSIVE CALLS)

- An execution of merge-sort is depicted by a binary tree
  - Each node represents a recursive call of merge-sort and stores
    - Unsorted sequence before the execution and its partition
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



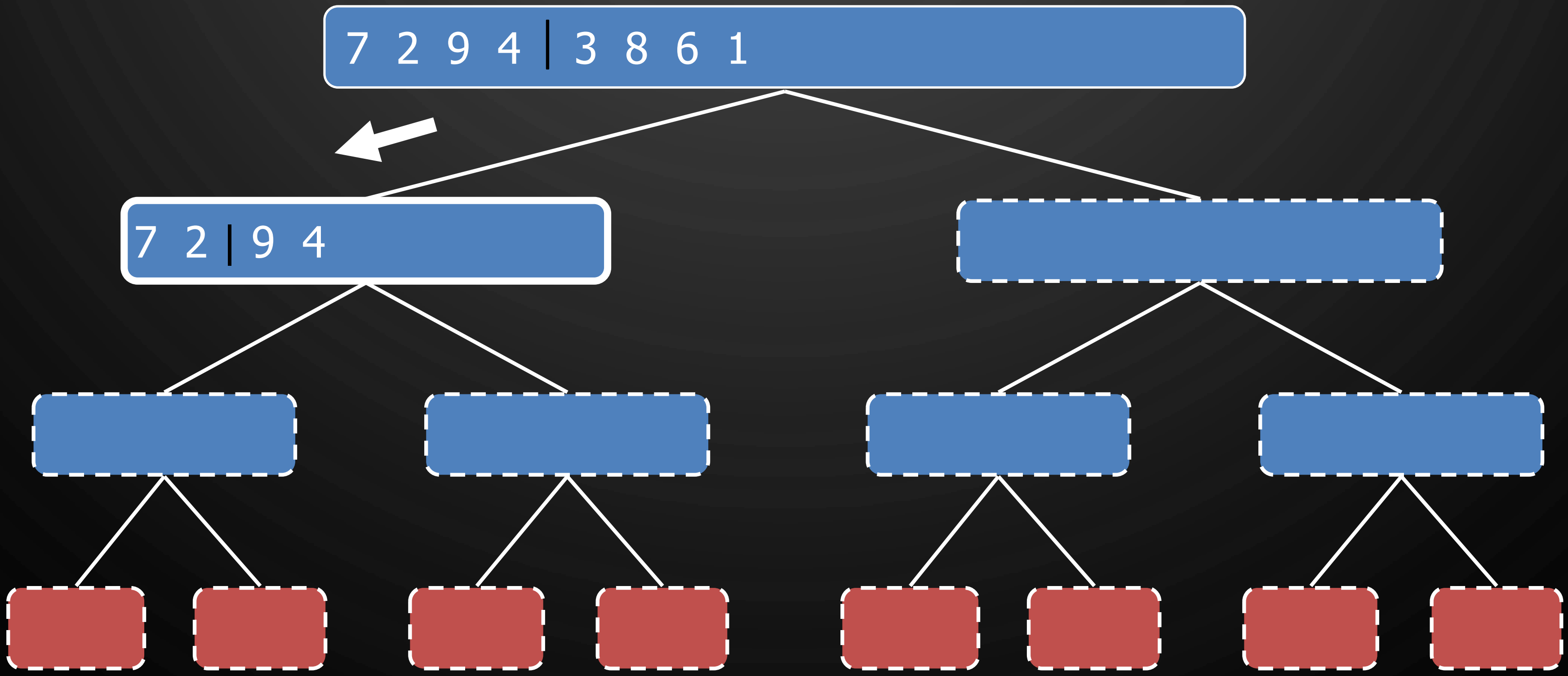
EXECUTION EXAMPLE

- Partition



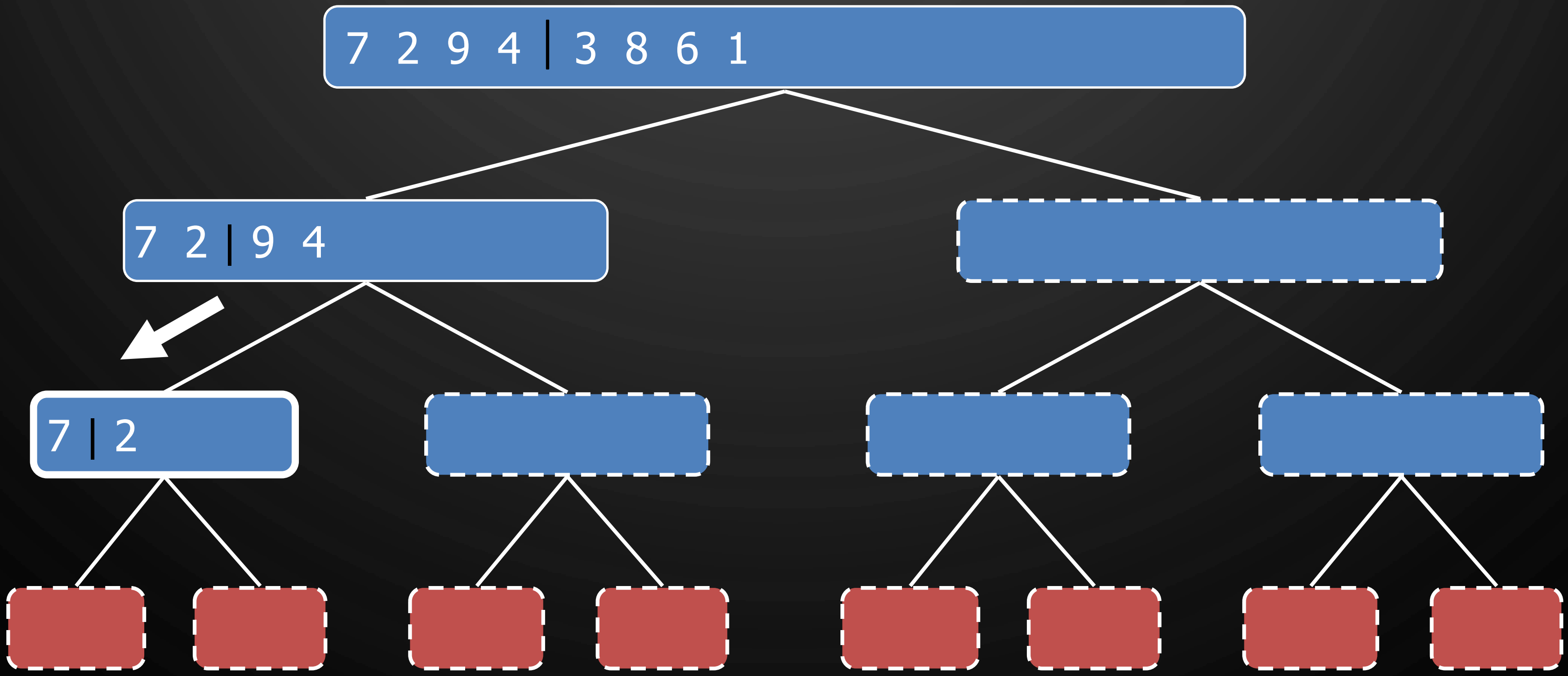
EXECUTION EXAMPLE

- Recursive Call, partition



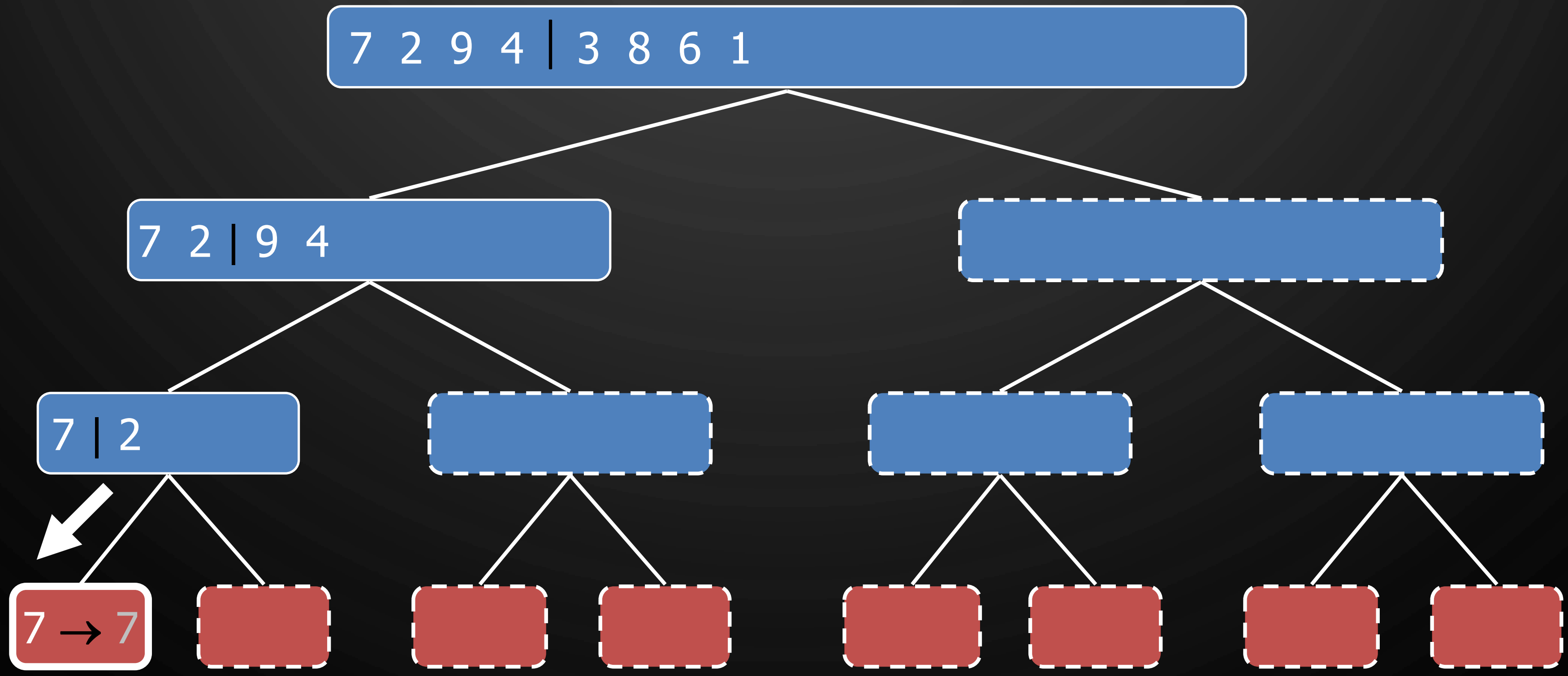
EXECUTION EXAMPLE

- Recursive Call, partition



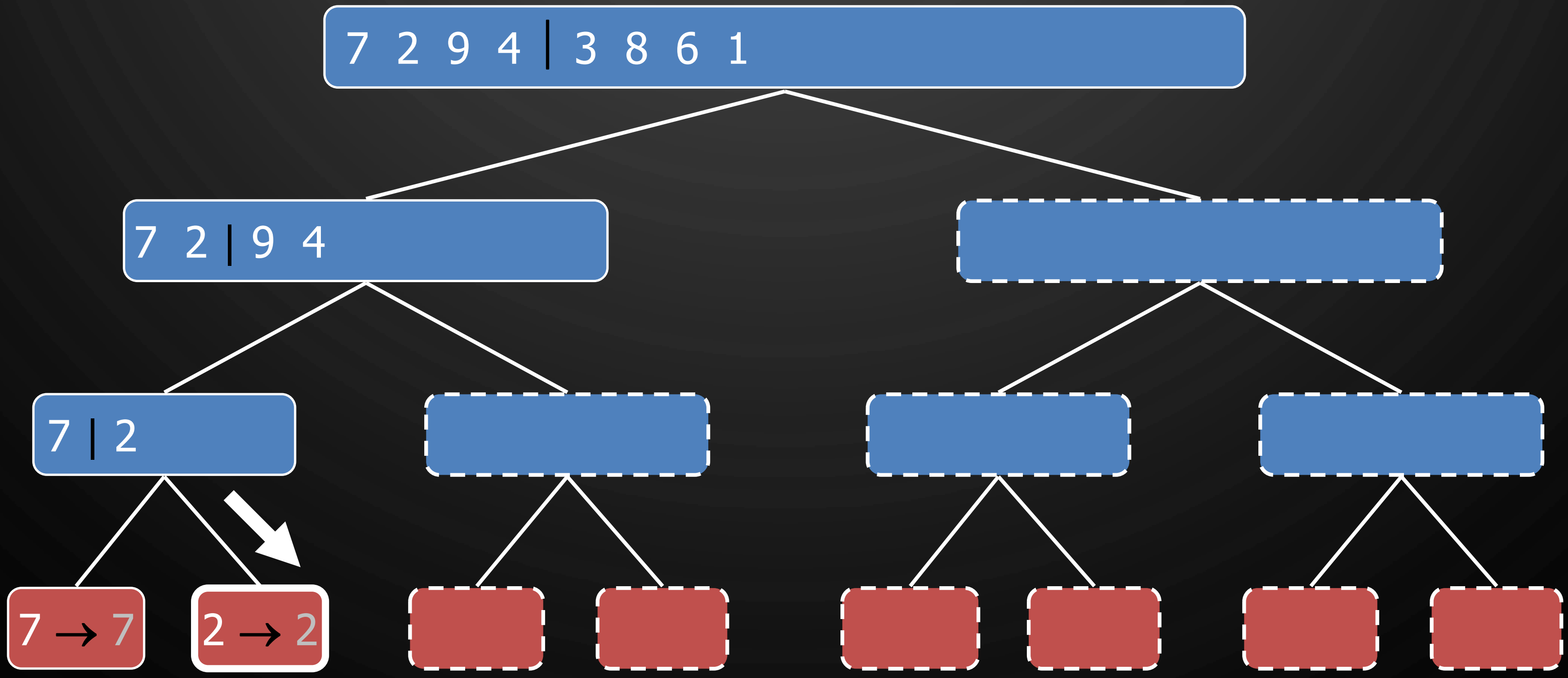
EXECUTION EXAMPLE

- Recursive Call, base case



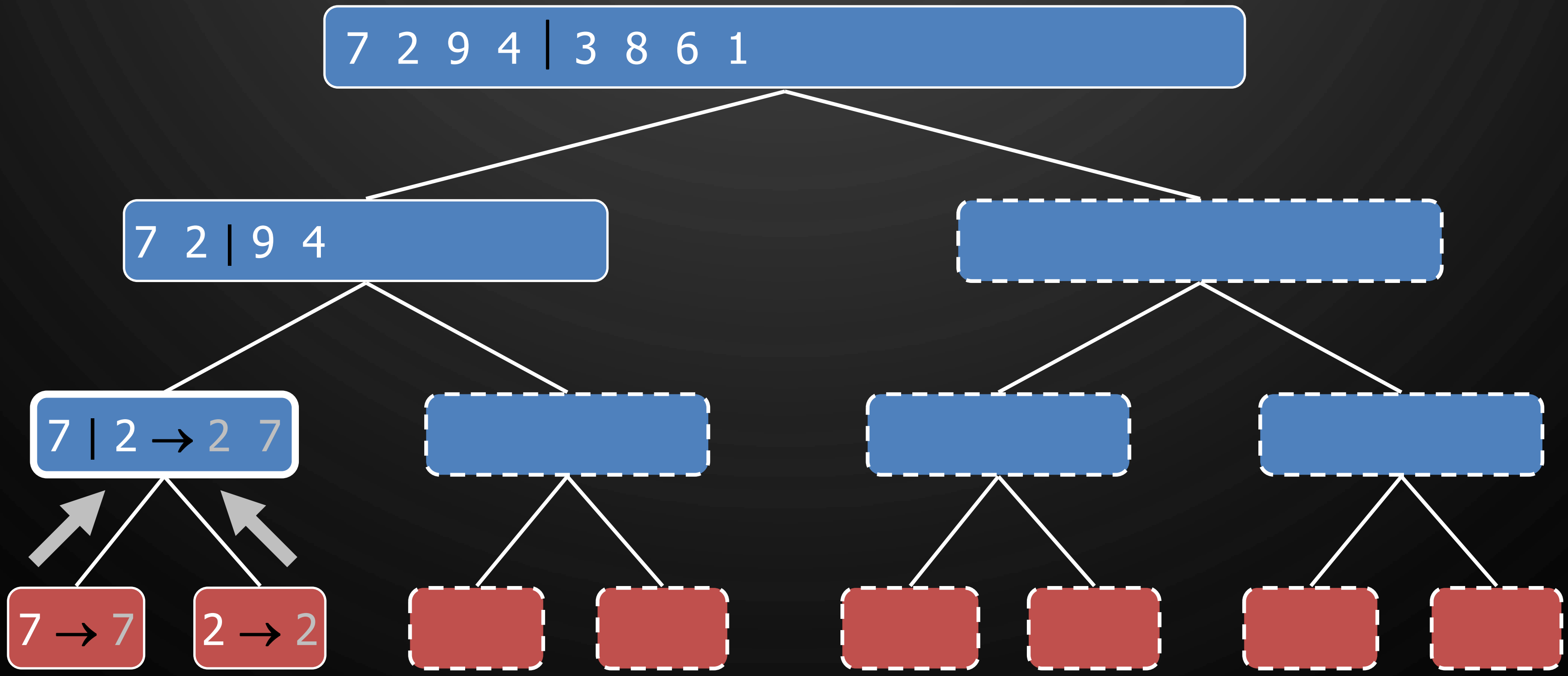
EXECUTION EXAMPLE

- Recursive Call, base case



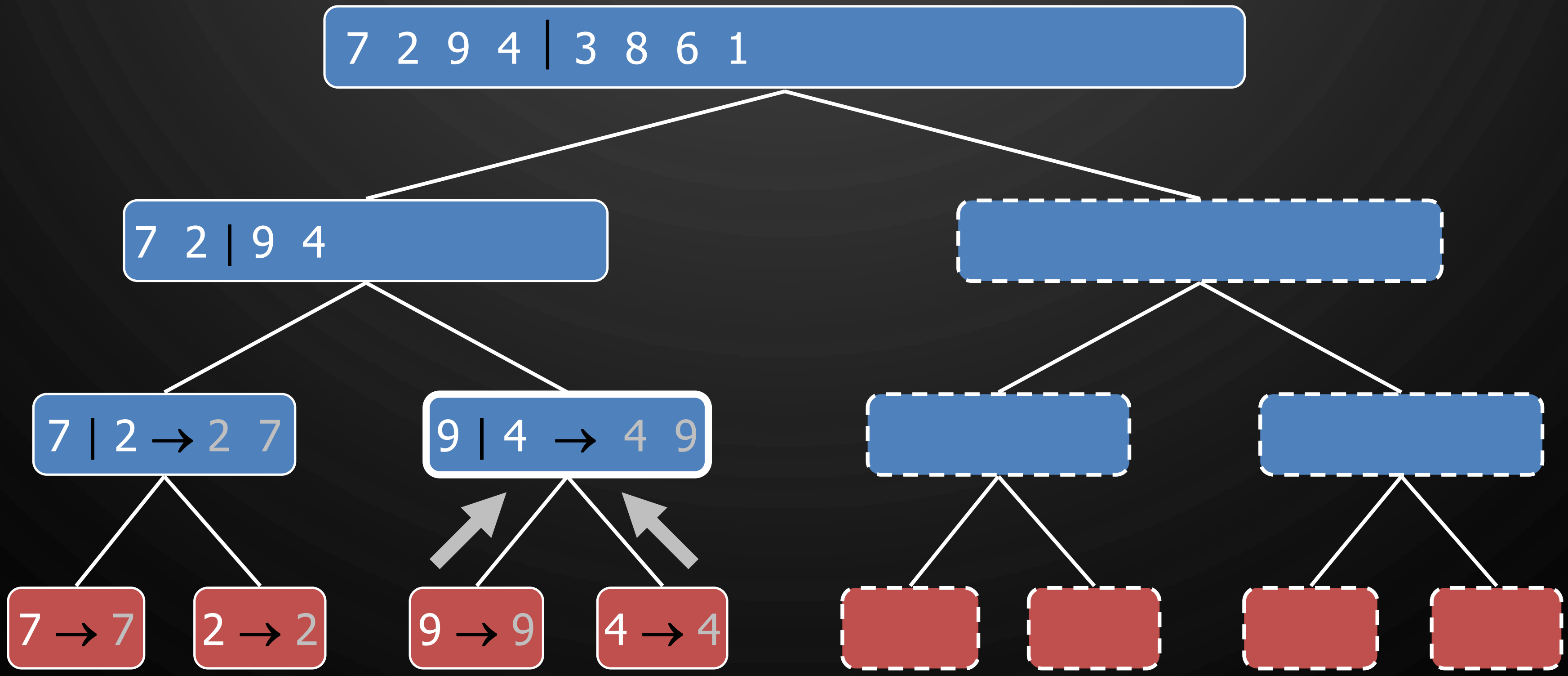
EXECUTION EXAMPLE

- Merge



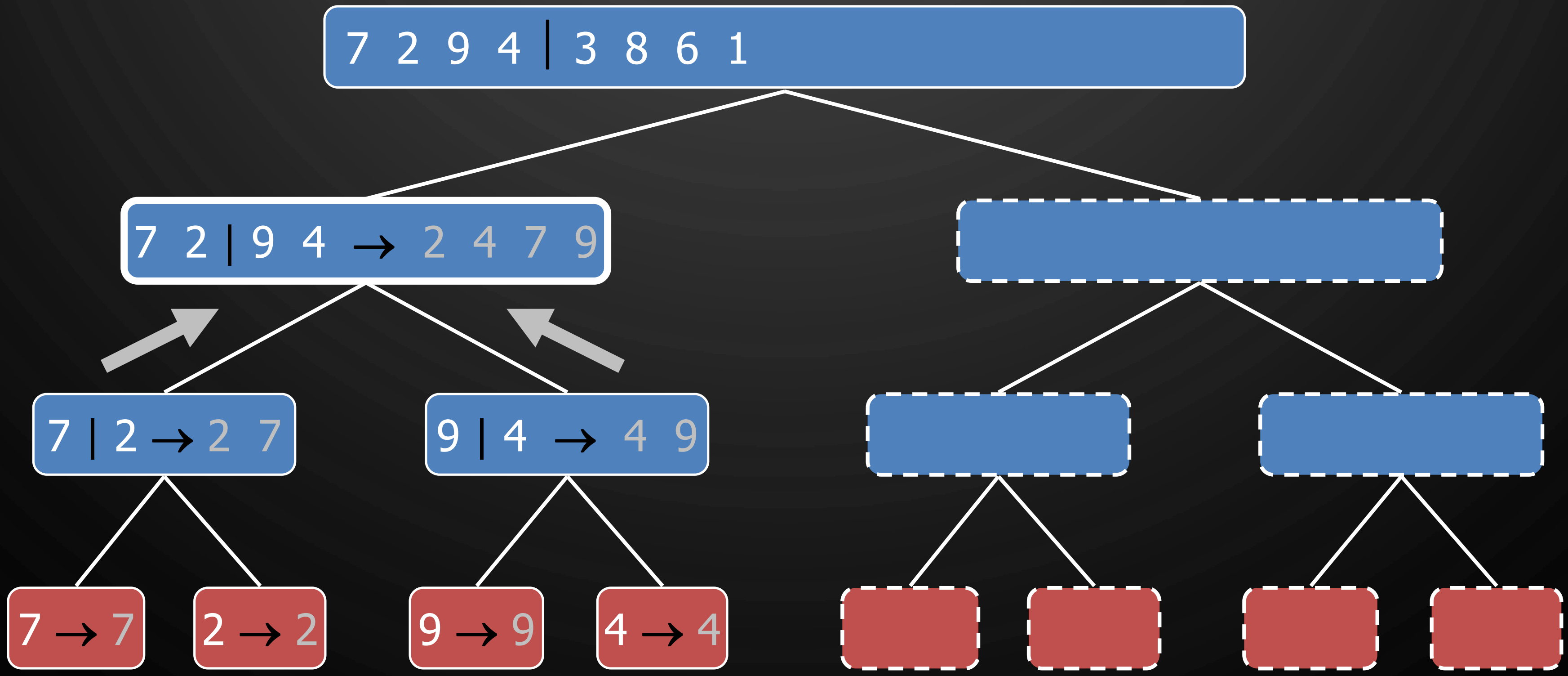
EXECUTION EXAMPLE

- Recursive call, …, base case, merge



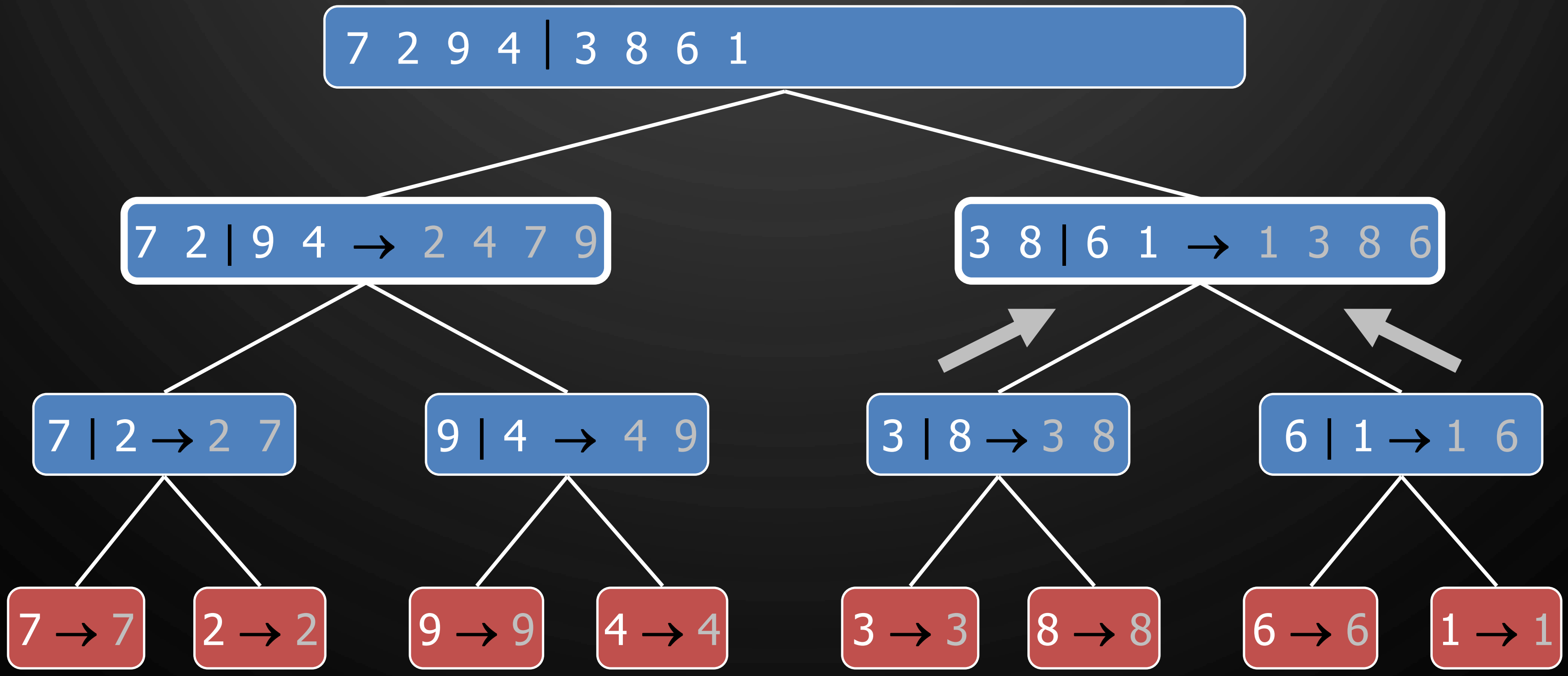
EXECUTION EXAMPLE

- Merge



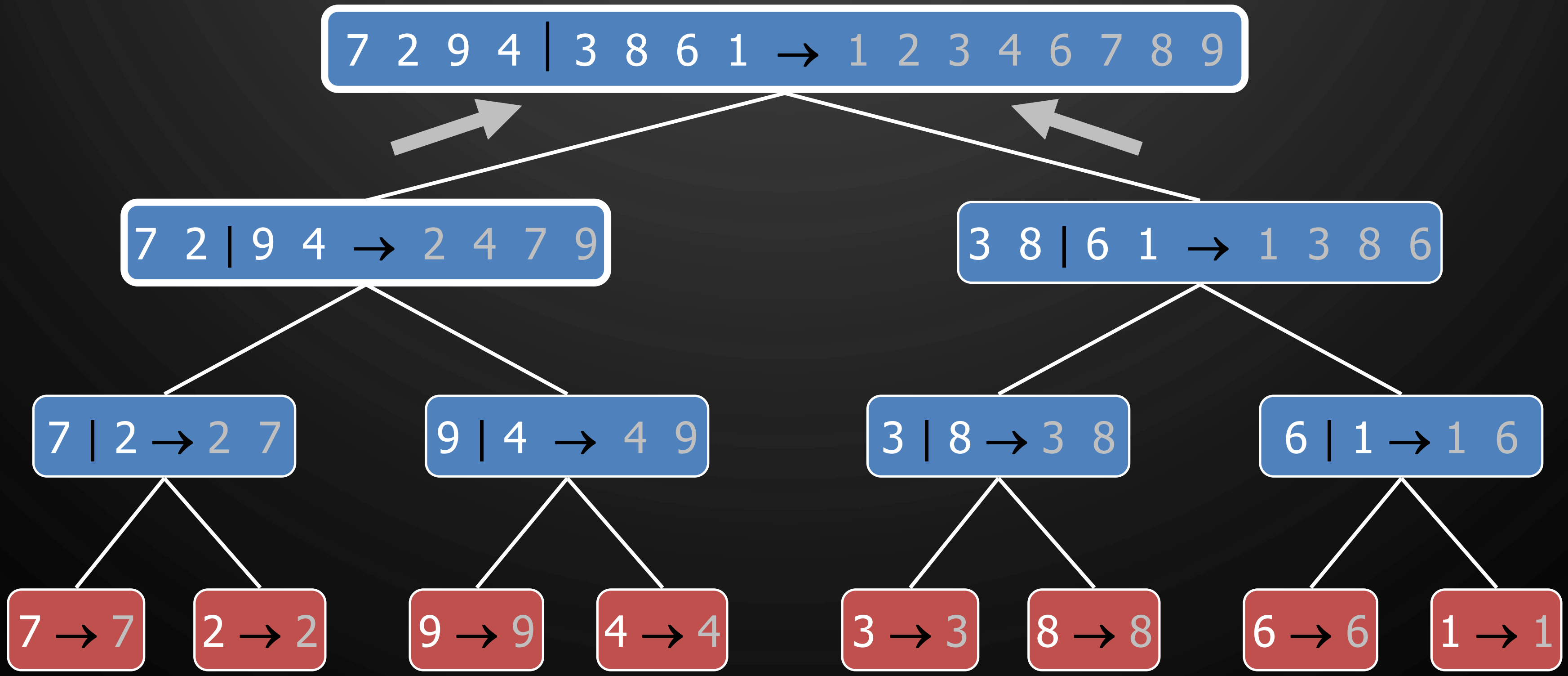
EXECUTION EXAMPLE

- Recursive call, …, merge, merge



EXECUTION EXAMPLE

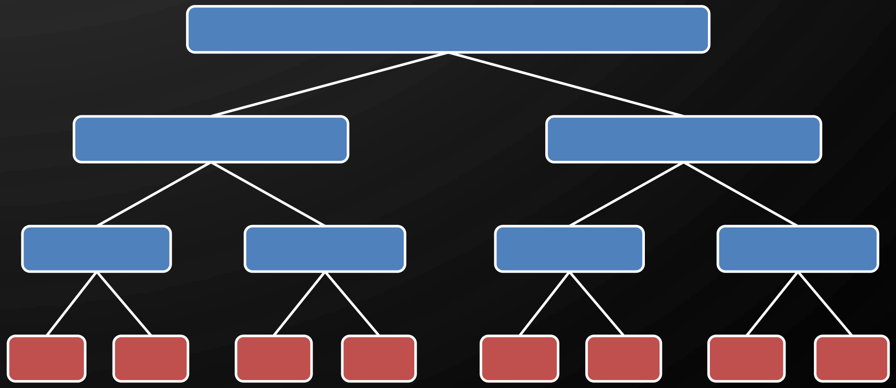
- Merge



ANOTHER ANALYSIS OF MERGE-SORT

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The work done at each level is $O(n)$
  - At level i, we partition and merge 2^i sequences of size $\frac{n}{2^i}$
- Thus, the total running time of merge-sort is $O(n \log n)$

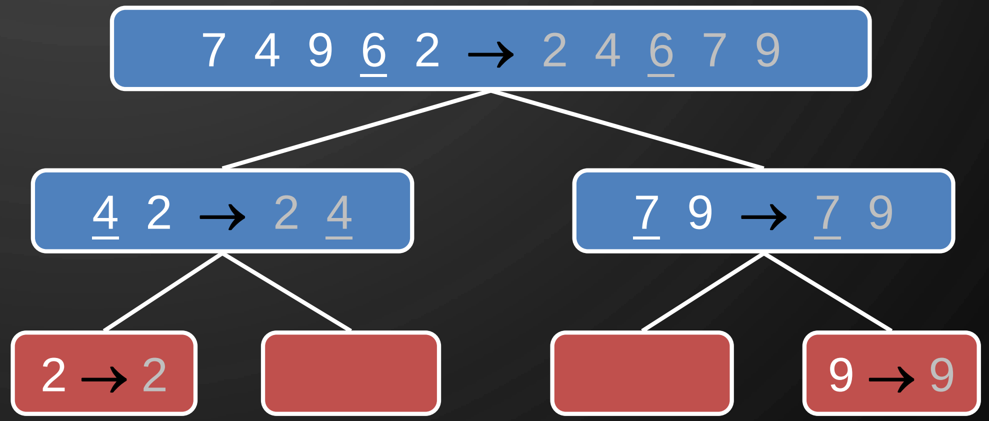
depth	#seqs	size	Cost for level
0	1	n	n
1	2	n/2	n
…	…	…	
i	2^i	$\frac{n}{2^i}$	n
…	…	…	
$\log n$	$2^{\log n} = n$	$\frac{n}{2^{\log n}} = 1$	n



SUMMARY OF SORTING ALGORITHMS (SO FAR)

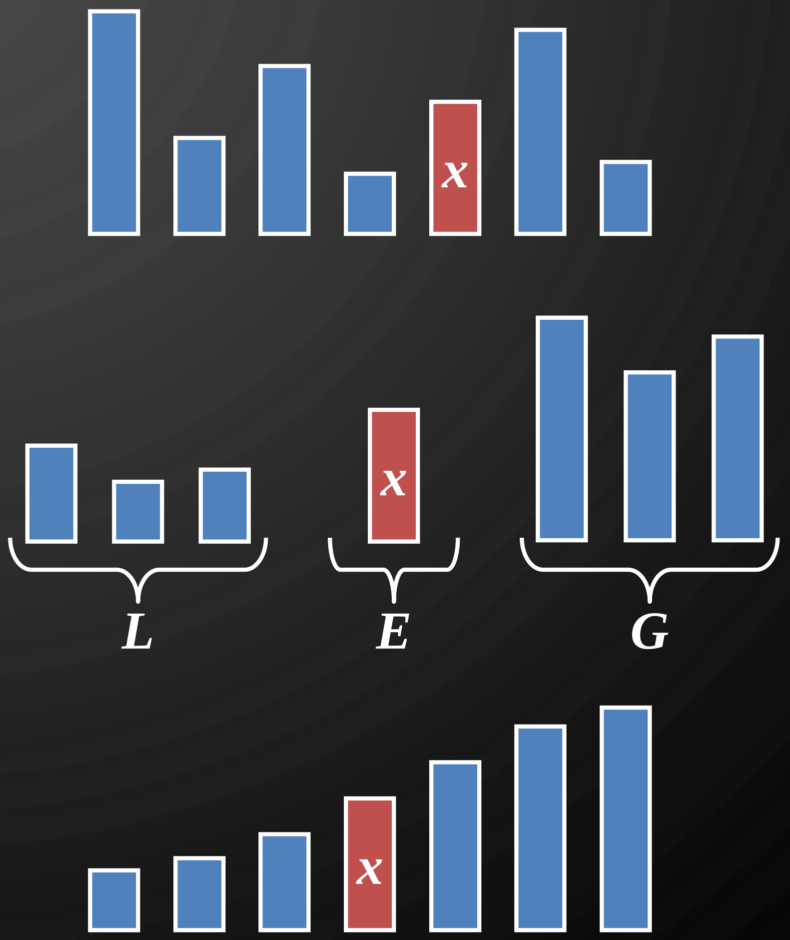
Algorithm	Time	Notes
Selection Sort	$O(n^2)$	Slow, in-place For small data sets
Insertion Sort	$O(n^2)$ WC, AC $O(n)$ BC	Slow, in-place For small data sets
Heap Sort	$O(n \log n)$	Fast, in-place For large data sets
Merge Sort	$O(n \log n)$	Fast, sequential data access For huge data sets

QUICK-SORT



QUICK-SORT

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L - elements less than x
    - E - elements equal x
    - G - elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E, and G



ANALYSIS OF QUICK SORT USING RECURRENCE RELATIONS

- Assumption: random pivot expected to give equal sized sublists
- The running time of Quick Sort can be expressed as:

$$T(n) = 2T\left(\frac{n}{2}\right) + P(n)$$

- $P(n)$ - time to run $\text{partition}(\quad)$ on input of size n

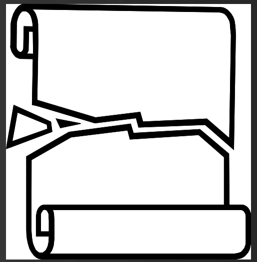
Algorithm $\text{quickSort}(S, l, r)$

Input: Sequence S, indices l, r

Output: Sequence S with the elements between l and r sorted

1. **if** $l \geq r$
2. **return** S
3. $i \leftarrow \text{rand}(\quad)\%(r - l) + l$ //random integer between l and r
4. $x \leftarrow S.\text{at}(i)$
5. $(h, k) \leftarrow \text{partition}(x)$
6. $\text{quickSort}(S, l, h - 1)$
7. $\text{quickSort}(S, k + 1, r)$
8. **return** S

PARTITION



- We partition an input sequence as follows:
    - We remove, in turn, each element y from S and
    - We insert y into L, E, or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

Algorithm $\text{partition}(S, p)$

Input: Sequence S, position p of the pivot

Output: Subsequences L, E, G of the elements of S less than, equal to, or greater than the pivot, respectively

$L, E, G \leftarrow \emptyset$
$x \leftarrow S.\text{erase}(p)$
while $\neg S.\text{empty}(\quad)$
    $y \leftarrow S.\text{eraseFront}(\quad)$
    if $y < x$
        $L.\text{insertBack}(y)$
    else if $y = x$
        $E.\text{insertBack}(y)$
    else $//y > x$
        $G.\text{insertBack}(y)$
return L, E, G

SO, THE EXPECTED COMPLEXITY OF QUICK SORT

- Assumption: random pivot expected to give equal sized sublists
- The running time of Quick Sort can be expressed as:

$$T(n) = 2T\left(\frac{n}{2}\right) + P(n)$$
$$= 2T\left(\frac{n}{2}\right) + O(n)$$
$$= O(n \log n)$$

Algorithm $\text{quickSort}(S, l, r)$

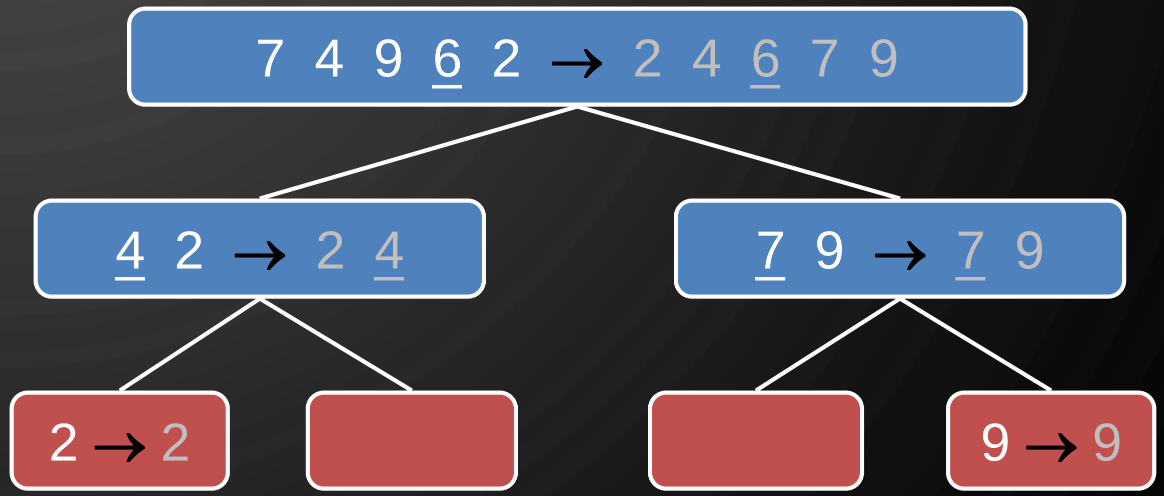
Input: Sequence S, indices l, r

Output: Sequence S with the elements between l and r sorted

1. **if** $l \geq r$
2. **return** S
3. $i \leftarrow \text{rand}(\quad)\%(r - l) + l$
   $//$random integer between l and r
4. $x \leftarrow S.\text{at}(i)$
5. $(h, k) \leftarrow \text{partition}(x)$
6. $\text{quickSort}(S, l, h - 1)$
7. $\text{quickSort}(S, k + 1\ , r)$
8. **return** S

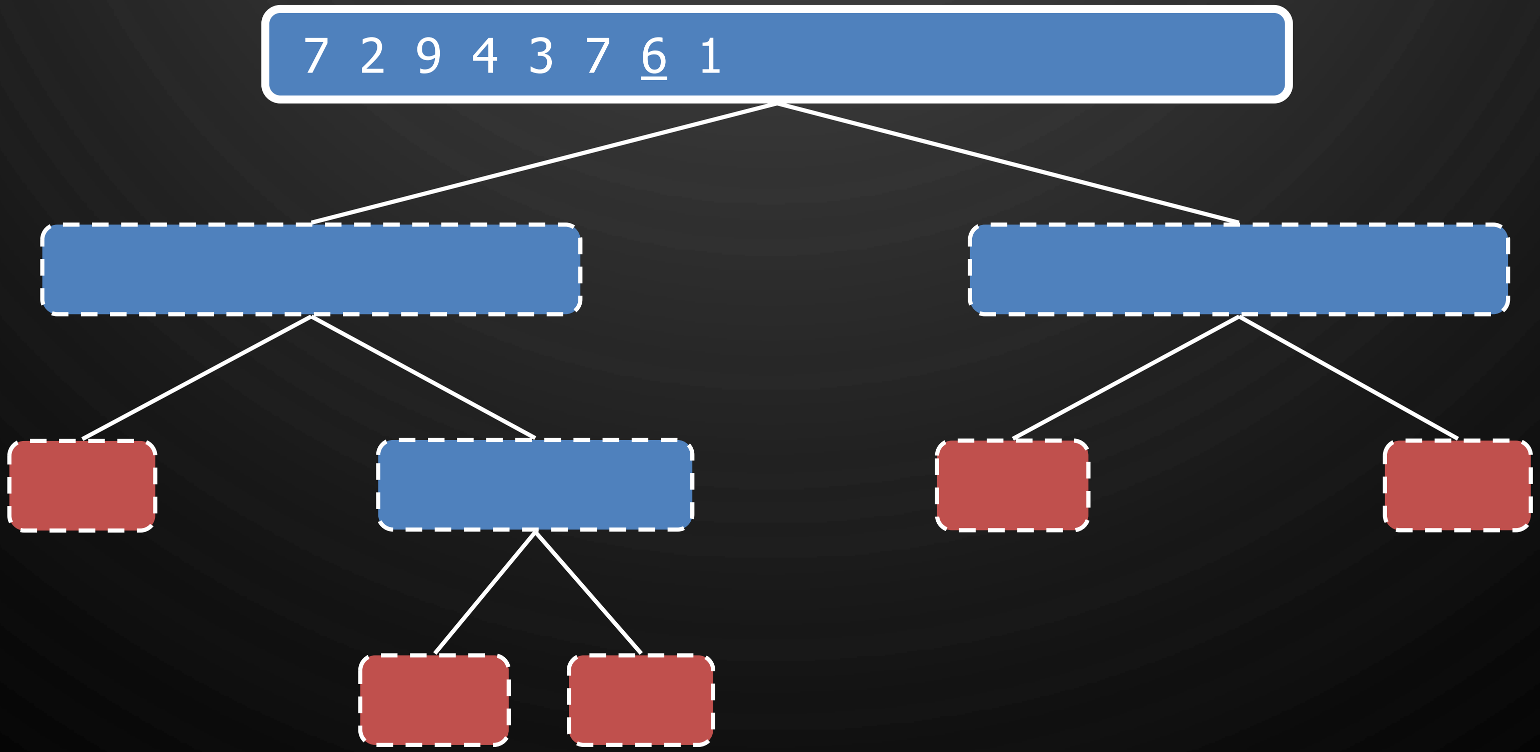
QUICK-SORT TREE

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



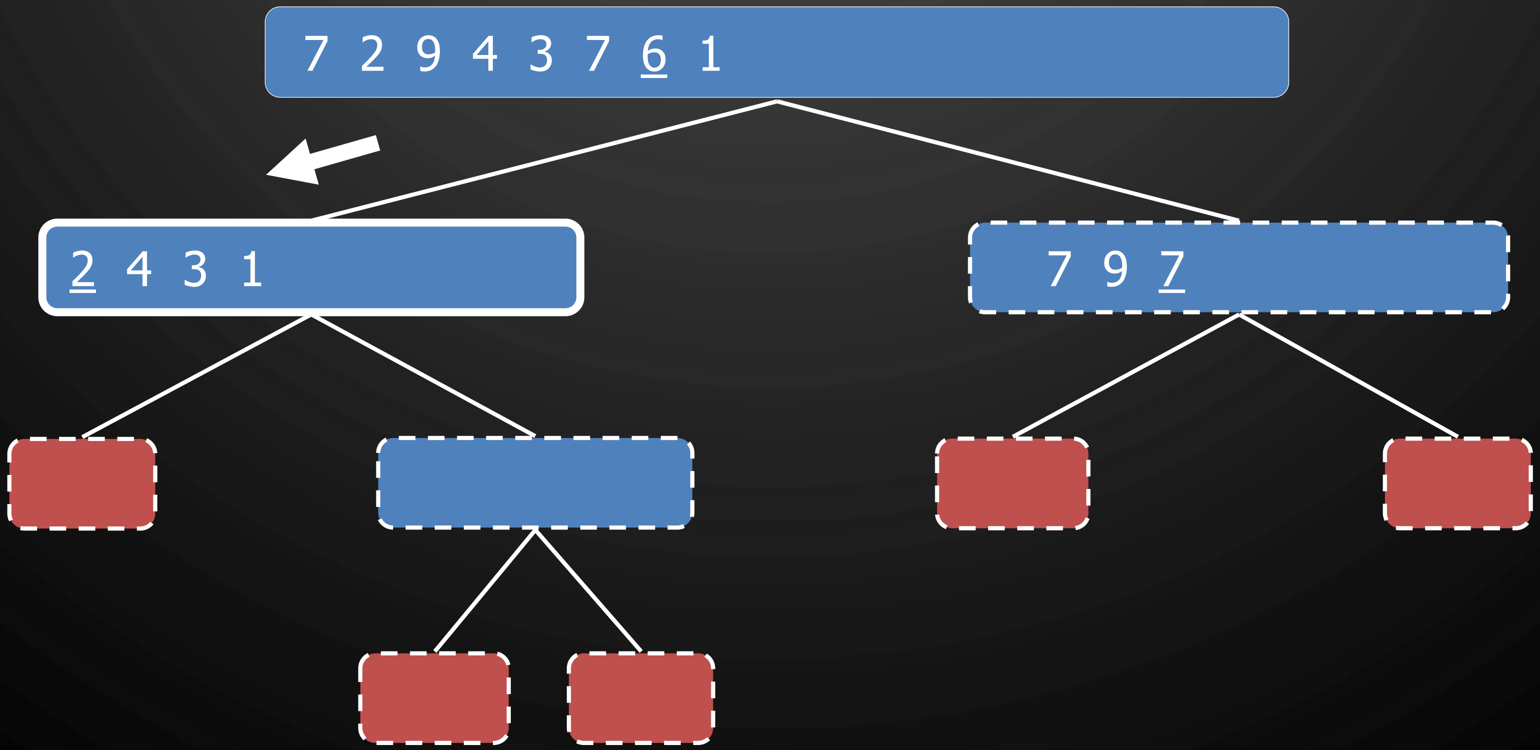
EXECUTION EXAMPLE

- Pivot selection



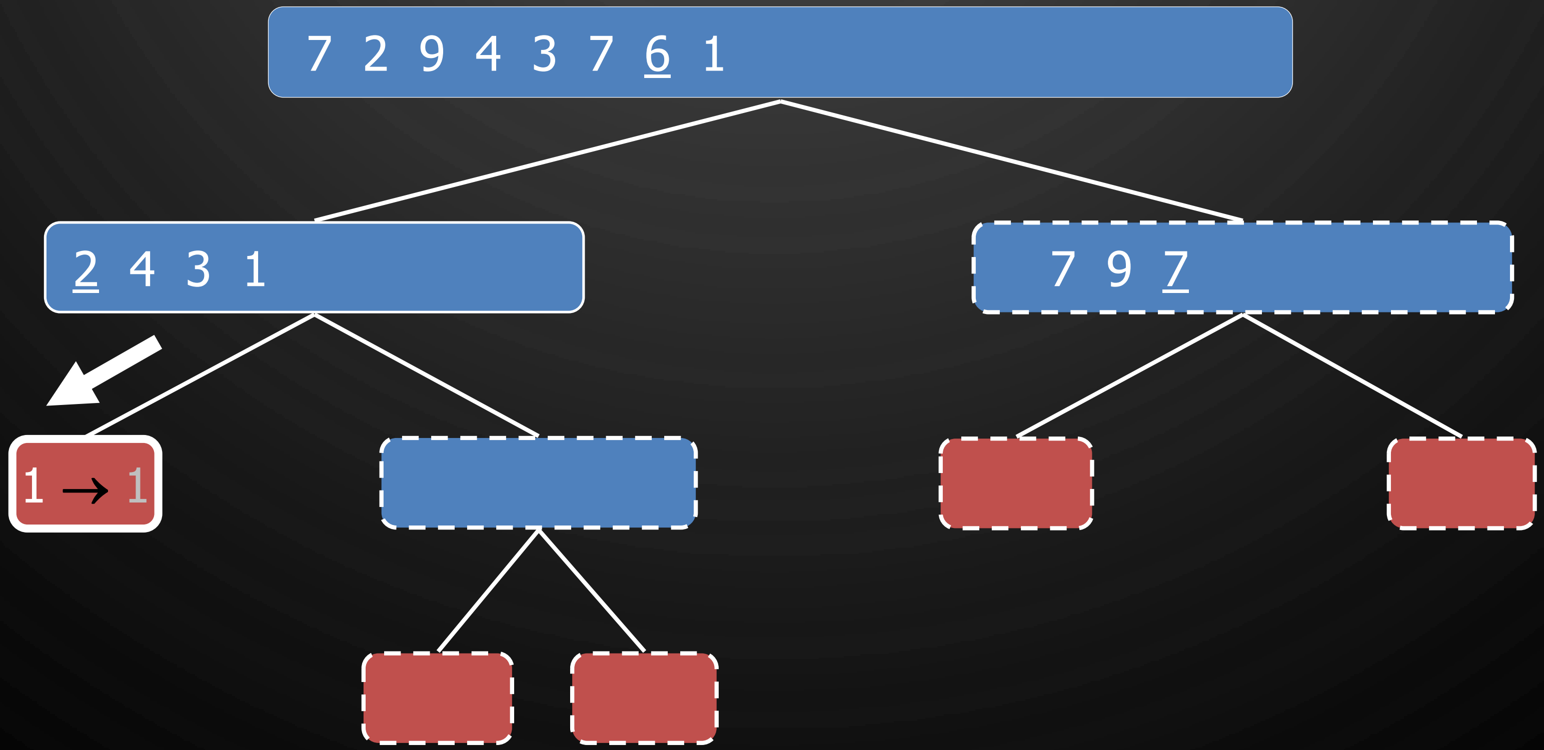
EXECUTION EXAMPLE

- Partition, recursive call, pivot selection



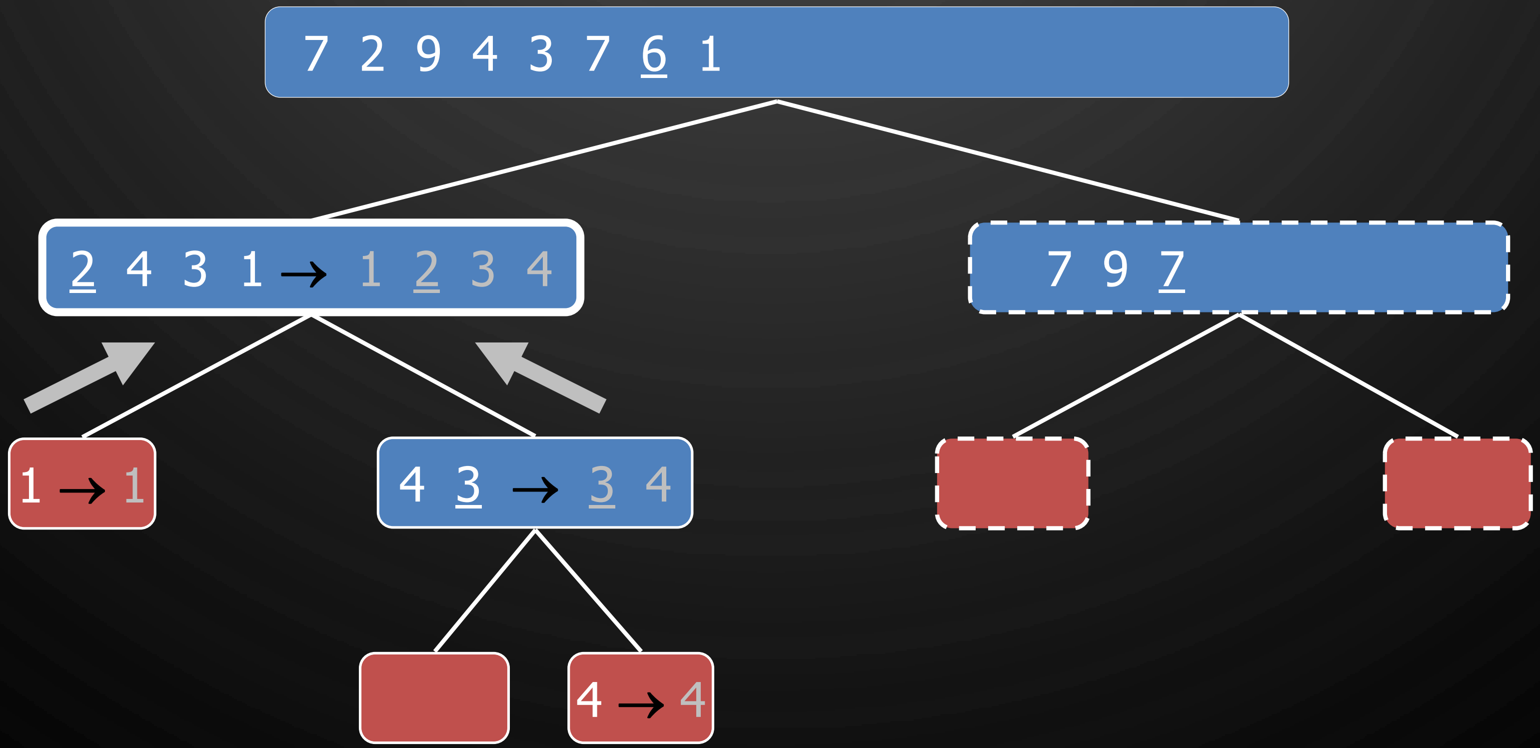
EXECUTION EXAMPLE

- Partition, recursive call, base case



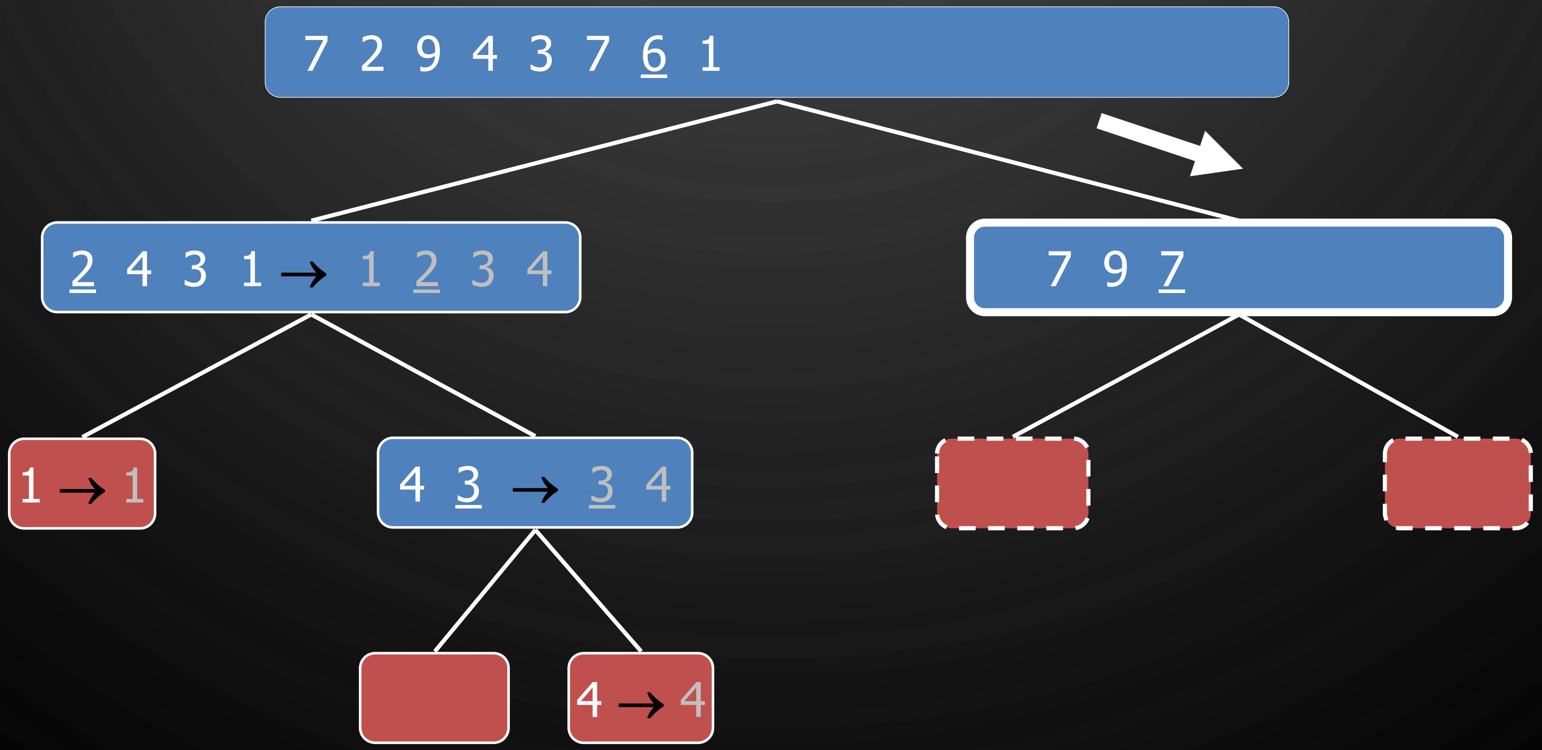
EXECUTION EXAMPLE

- Recursive call, …, base case, join



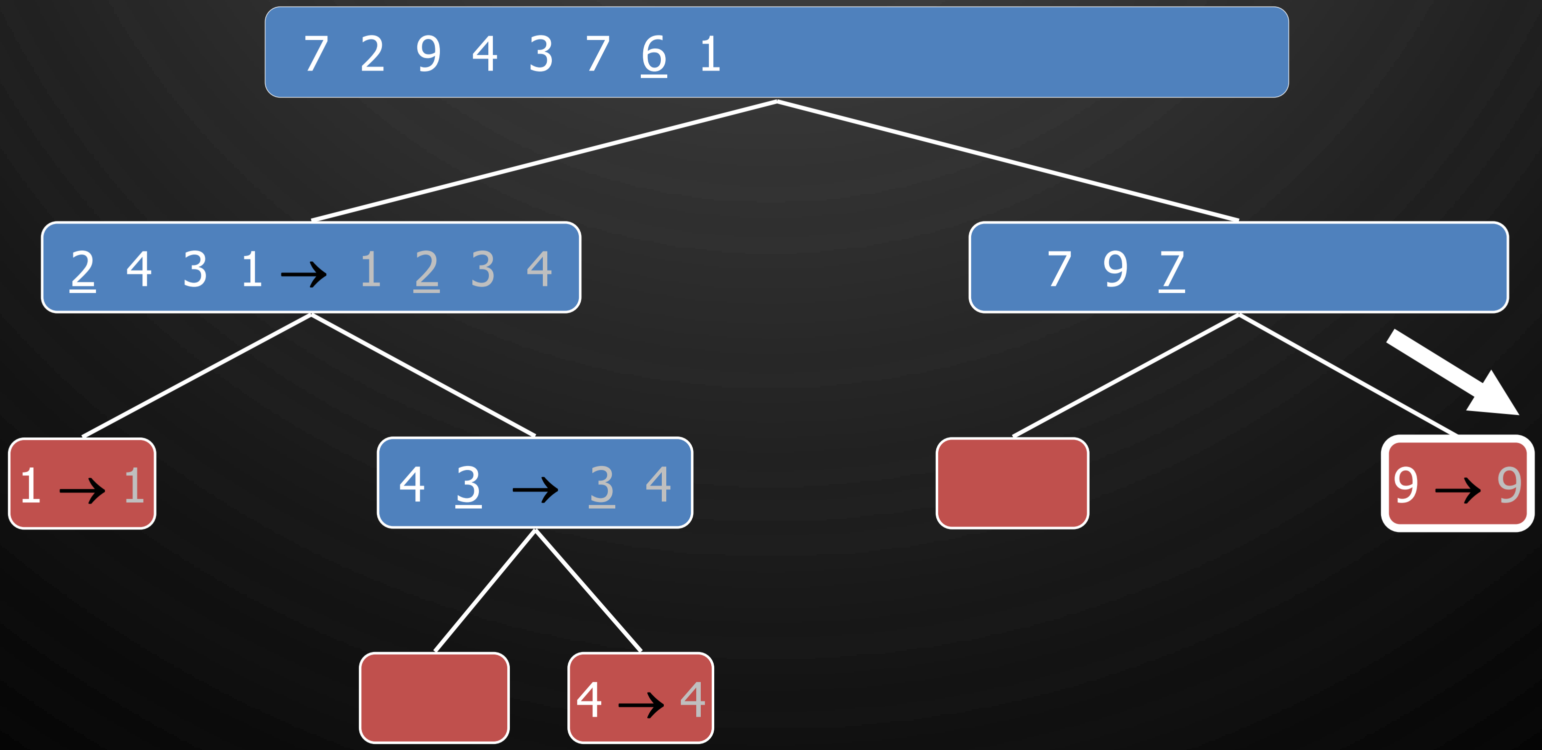
EXECUTION EXAMPLE

- Recursive call, pivot selection



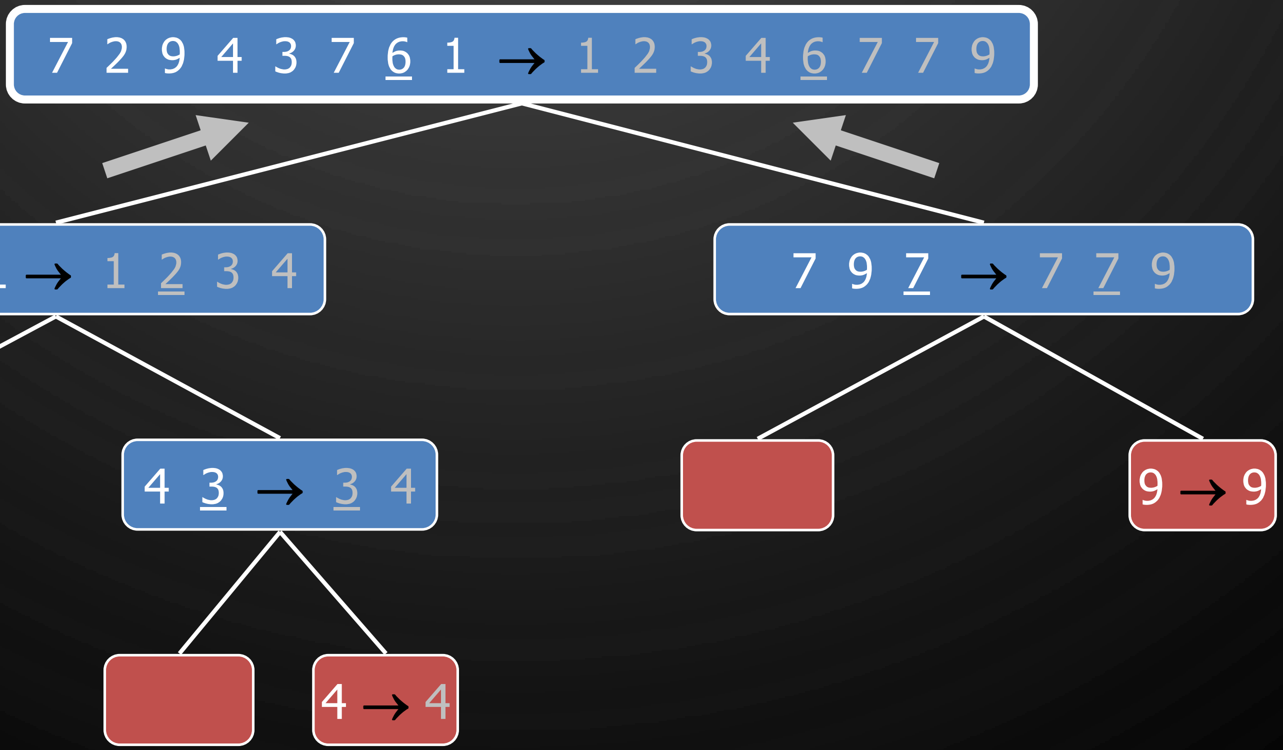
EXECUTION EXAMPLE

- Partition, …, recursive call, base case



EXECUTION EXAMPLE

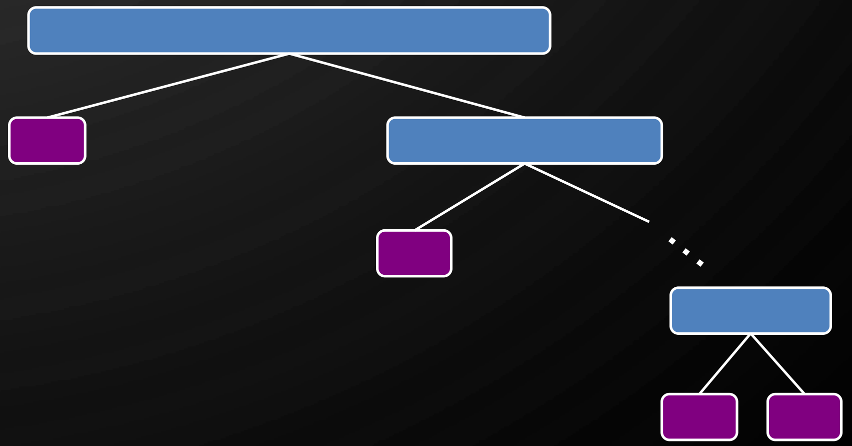
- Join, join



WORST-CASE RUNNING TIME

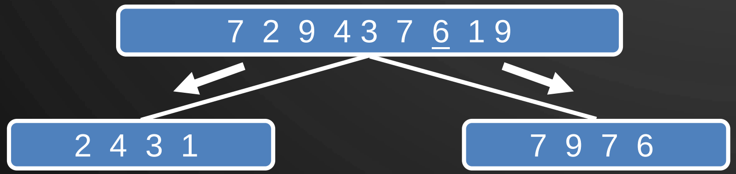
- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
  - One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to:
  $$n + (n-1) + \cdots + 2 + 1 = O(n^2)$$
- Alternatively, using recurrence equations:
  $$T(n) = T(n-1) + O(n) = O(n^2)$$

depth	time
0	n
1	$n - 1$
…	…
$n - 1$	1

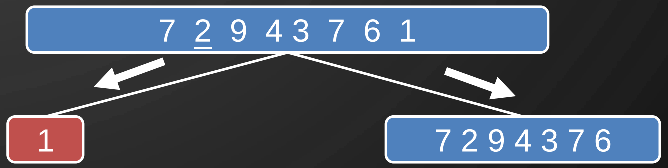


EXPECTED RUNNING TIME
REMOVING EQUAL SPLIT ASSUMPTION

- Consider a recursive call of quick-sort on a sequence of size s
  - Good call: the sizes of L and G are each less than $\frac{3s}{4}$
  - Bad call: one of L and G has size greater than $\frac{3s}{4}$

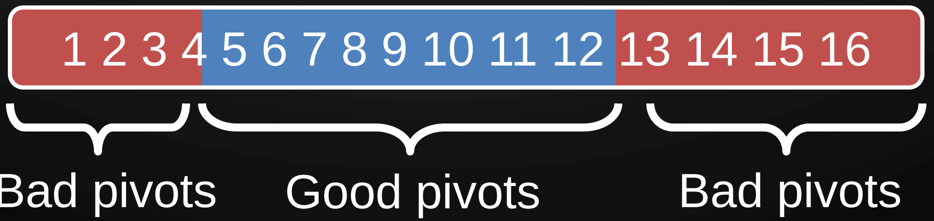


Good call



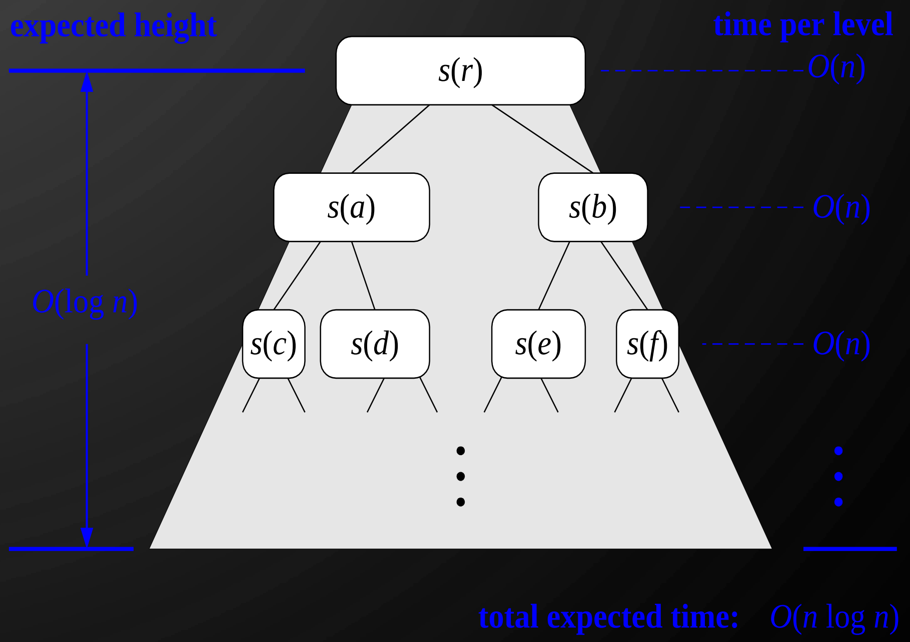
Bad call

- A call is good with probability $1/2$
  - $1/2$ of the possible pivots cause good calls:



EXPECTED RUNNING TIME

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$ (e.g., it is expected to take 2 tosses to get heads)
- For a node of depth i, we expect
  - $\frac{i}{2}$ ancestors are good calls
  - The size of the input sequence for the current call is at most $\left(\frac{3}{4}\right)^{\frac{i}{2}} n$
- Therefore, we have
  - For a node of depth $2\log_{\frac{4}{3}} n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$



IN-PLACE QUICK-SORT

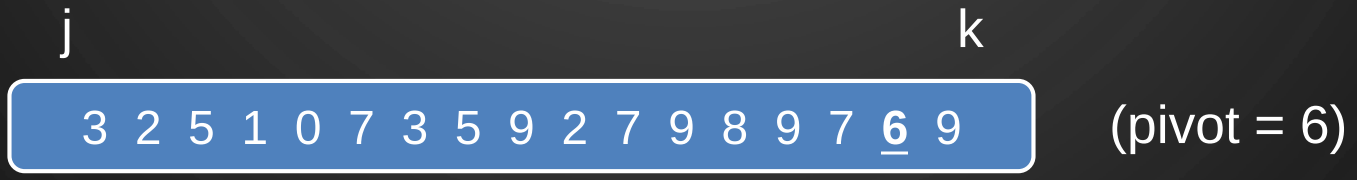
- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have indices less than h
  - the elements equal to the pivot have indices between h and k
  - the elements greater than the pivot have indices greater than k
- The recursive calls consider
  - elements with indices less than h
  - elements with indices greater than k

Algorithm $\mathrm{inPlaceQuickSort}(S, l, r)$
Input: Array S, indices l, r
Output: Array S with the elements between l and r sorted

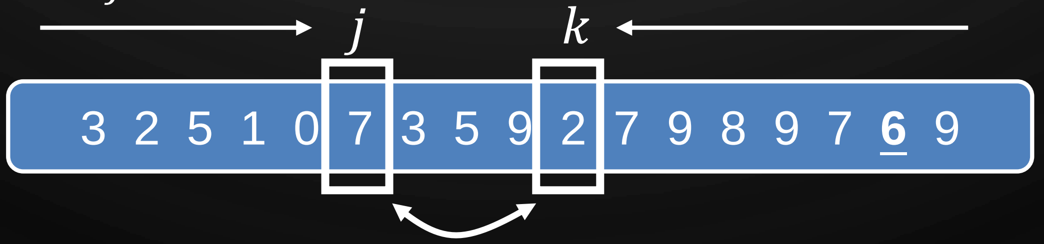
1. **if** $l \geq r$
2. **return** S
3. $i \leftarrow \mathrm{rand}(\quad)\%(r - l) + l$
   //random integer between l and r
4. $x \leftarrow S[i]$
5. $(h, k) \leftarrow \mathrm{inPlacePartition}(x)$
6. $\mathrm{inPlaceQuickSort}(S, l, h - 1)$
7. $\mathrm{inPlaceQuickSort}(S, k + 1, r)$
8. **return** S

IN-PLACE PARTITIONING

- Perform the partition using two indices to split S into L and $E \cup G$ (a similar method can split $E \cup G$ into E and G).



- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq x$.
  - Scan k to the left until finding an element $< x$.
  - Swap elements at indices j and k



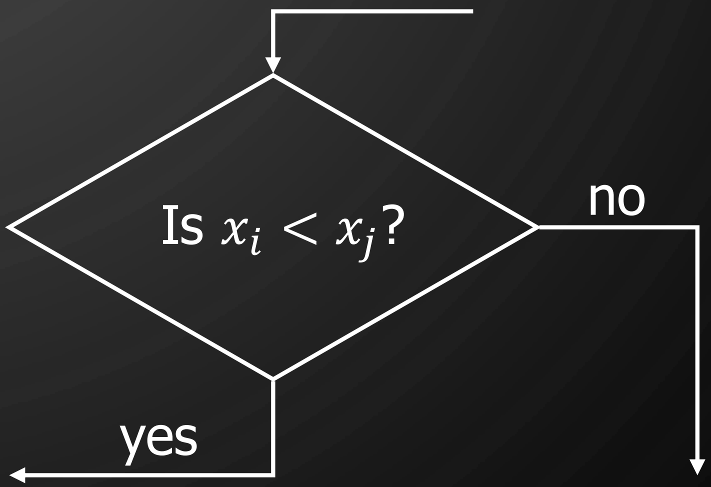
SUMMARY OF SORTING ALGORITHMS (SO FAR)

Algorithm	Time	Notes
Selection Sort	$O(n^2)$	Slow, in-place For small data sets
Insertion Sort	$O(n^2)$ WC, AC $O(n)$ BC	Slow, in-place For small data sets
Heap Sort	$O(n \log n)$	Fast, in-place For large data sets
Quick Sort	Exp. $O(n \log n)$ AC, BC $O(n^2)$ WC	Fastest, randomized, in-place For large data sets
Merge Sort	$O(n \log n)$	Fast, sequential data access For huge data sets

SORTING LOWER BOUND

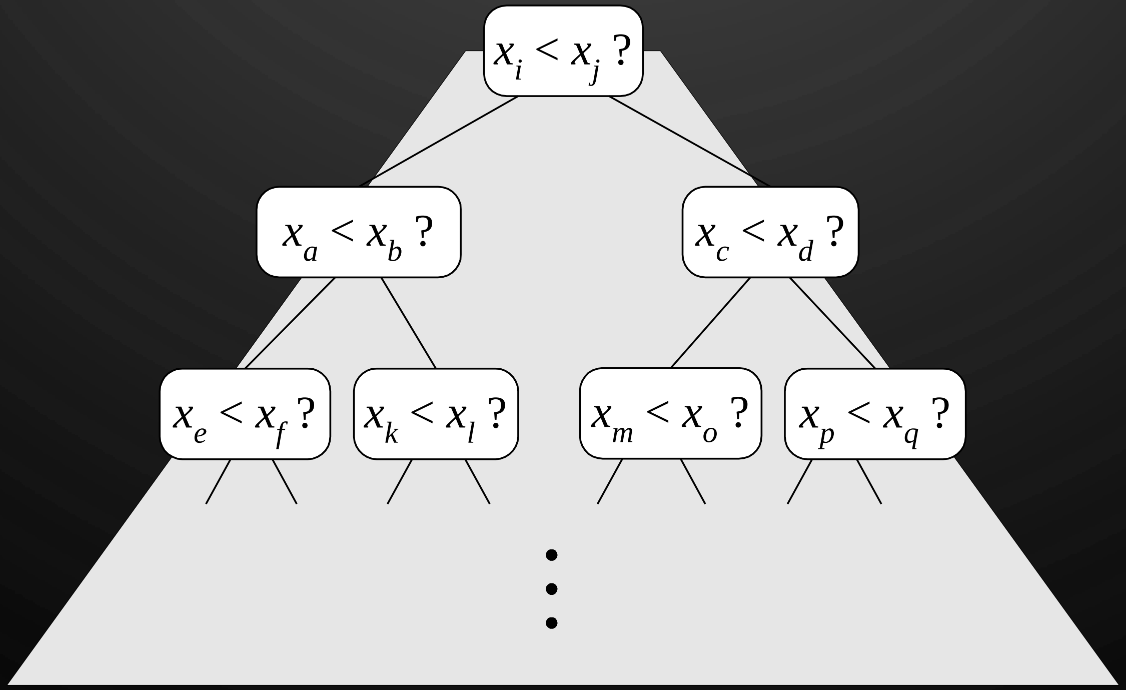
COMPARISON-BASED SORTING

- Many sorting algorithms are comparison based.
  - They sort by making comparisons between pairs of objects
  - Examples: bubble-sort, selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, …
- Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements, $x_1, x_2, \ldots, x_n$.



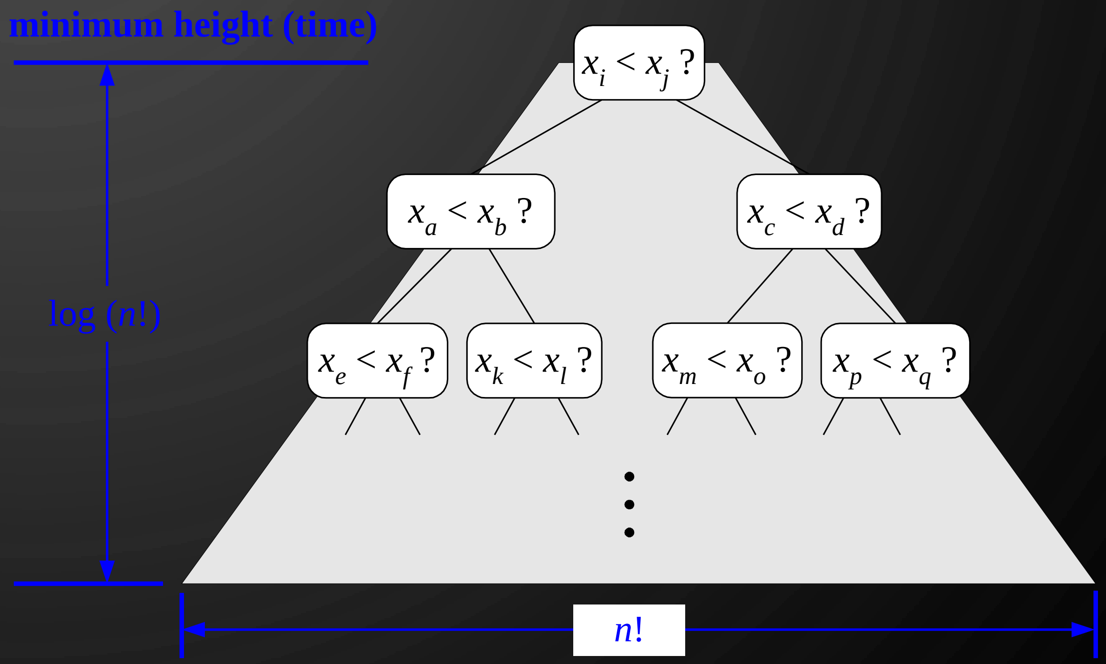
COUNTING COMPARISONS

- Let us just count comparisons then.
- Each possible run of the algorithm corresponds to a root-to-leaf path in a decision tree

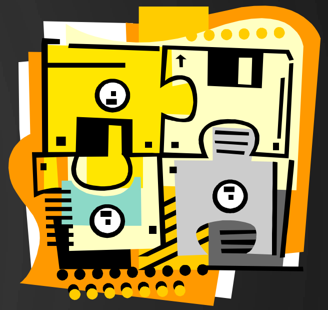


DECISION TREE HEIGHT

- The height of the decision tree is a lower bound on the running time
- Every input permutation must lead to a separate leaf output
- If not, some input …4…5… would have same output ordering as …5…4…, which would be wrong
- Since there are $n! = 1 * 2 * \cdots * n$ leaves, the height is at least $\log(n!)$



THE LOWER BOUND



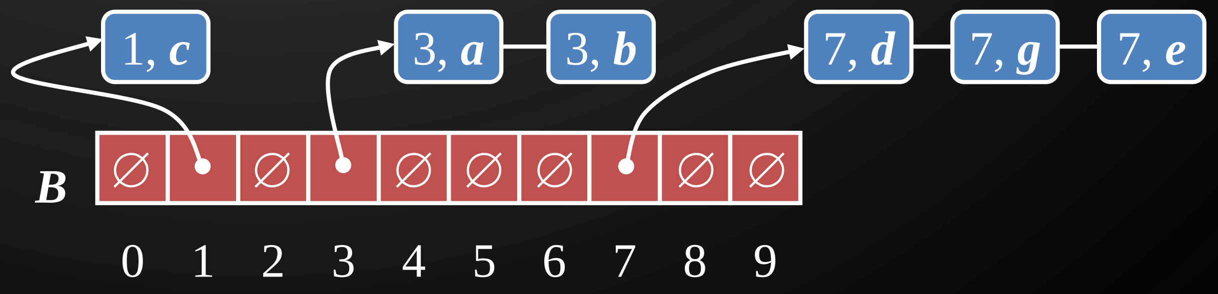
- Any comparison-based sorting algorithm takes at least $\log(n!)$ time

$$\log(n!) \geq \log\left(\frac{n}{2}\right)^{\frac{n}{2}} = \frac{n}{2}\log\frac{n}{2}$$

- That is, any comparison-based sorting algorithm must run in $\Omega(n \log n)$ time.

BUCKET-SORT AND RADIX-SORT

CAN WE SORT IN LINEAR TIME?



BUCKET-SORT

- Let be S be a sequence of n (key, element) items with keys in the range $[0, N-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)
  - Phase 1: Empty sequence S by moving each entry into its bucket $B[k]$
  - Phase 2: for $i \leftarrow 0 \ldots N-1$, move the items of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n+N)$ time
- Bucket-sort takes $O(n+N)$ time

Algorithm $\text{bucketSort}(S, N)$
Input: Sequence S of entries with integer keys in the range $[0, N-1]$
Output: Sequence S sorted in nondecreasing order of the keys

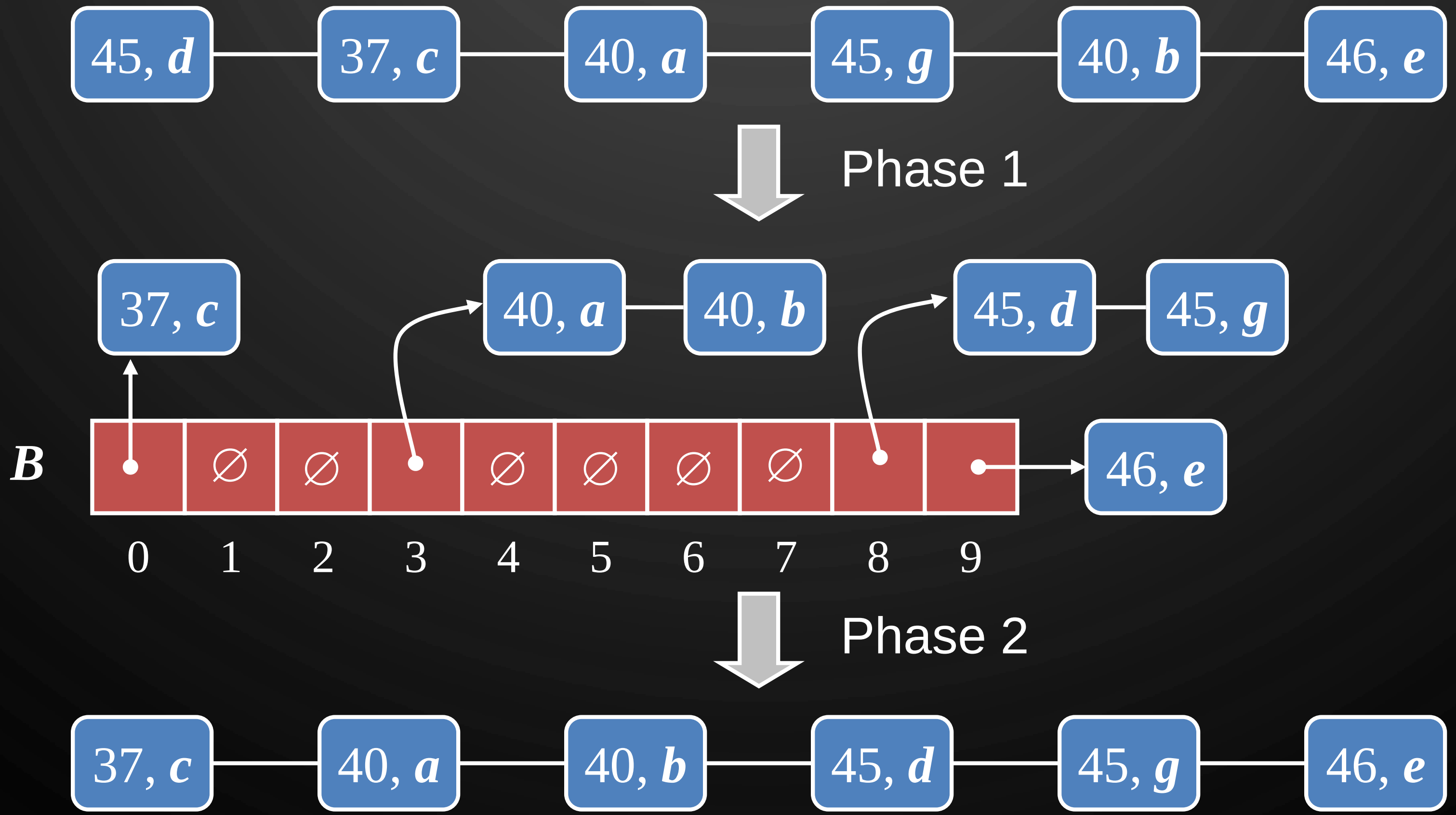
```
B ← array of N empty sequences
for each entry e ∈ S do
    k ← e.key( )
    remove e from S and insert it at the end of bucket B[k]
for i ← 0 … N − 1 do
    for each entry e ∈ B[i] do
        remove e from bucket B[i] and insert it at the end of S
```

PROPERTIES AND EXTENSIONS

- Properties
  - Key-type
    - The keys are used as indices into an array and cannot be arbitrary objects
    - No external comparator
  - Stable sorting
    - The relative order of any two items with the same key is preserved after the execution of the algorithm
- Extensions
  - Integer keys in the range $[a, b]$
    - Put entry e into bucket $B[k - a]$
  - String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
    - Sort D and compute the index $i(k)$ of each string k of D in the sorted sequence
    - Put item e into bucket $B[i(k)]$

EXAMPLE

- Key range [37, 46] – map to buckets [0,9]



LEXICOGRAPHIC ORDER

- Given a list of tuples:
  (7,4,6) (5,1,5) (2,4,6) (2,1,4) (5,1,6) (3,2,4)

- After sorting, the list is in lexicographical order:
  (2,1,4) (2,4,6) (3,2,4) (5,1,5) (5,1,6) (7,4,6)

LEXICOGRAPHIC ORDER FORMALIZED

- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
  - Example - the Cartesian coordinates of a point in space is a 3-tuple (x, y, z)
- The lexicographic order of two d-tuples is recursively defined as follows
- $(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d) \Leftrightarrow$

$$x_1 < y_1 \vee \left(x_1 = y_1 \wedge (x_2, \ldots, x_d) < (y_2, \ldots, y_d)\right)$$

- i.e., the tuples are compared by the first dimension, then by the second dimension, etc.

EXERCISE
LEXICOGRAPHIC ORDER

- Given a list of 2-tuples, we can order the tuples lexicographically by applying a stable sorting algorithm two times:
  (3,3) (1,5) (2,5) (1,2) (2,3) (1,7) (3,2) (2,2)
- Possible ways of doing it:
  - Sort first by 1st element of tuple and then by 2nd element of tuple
  - Sort first by 2nd element of tuple and then by 1st element of tuple
- Show the result of sorting the list using both options

EXERCISE
LEXICOGRAPHIC ORDER

- (3,3) (1,5) (2,5) (1,2) (2,3) (1,7) (3,2) (2,2)
- Using a stable sort,
  - Sort first by 1st element of tuple and then by 2nd element of tuple
  - Sort first by 2nd element of tuple and then by 1st element of tuple
- Option 1:
  - 1st sort: (1,5) (1,2) (1,7) (2,5) (2,3) (2,2) (3,3) (3,2)
  - 2nd sort: (1,2) (2,2) (3,2) (2,3) (3,3) (1,5) (2,5) (1,7) - WRONG
- Option 2:
  - 1st sort: (1,2) (3,2) (2,2) (3,3) (2,3) (1,5) (2,5) (1,7)
  - 2nd sort: (1,2) (1,5) (1,7) (2,2) (2,3) (2,5) (3,2) (3,3) - CORRECT

LEXICOGRAPHIC-SORT

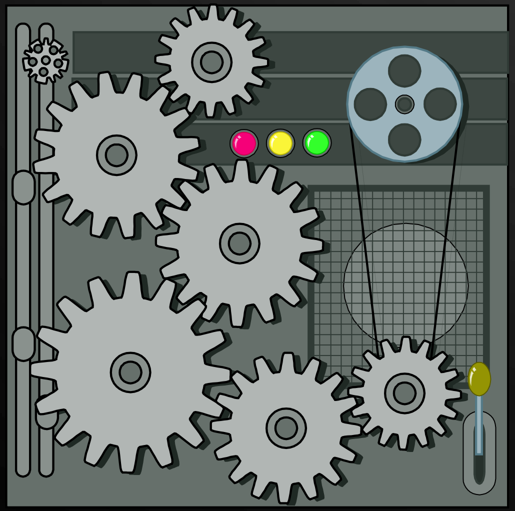
- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let $\text{stableSort}(S, C)$ be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm stableSort, one per dimension
- Lexicographic-sort runs in $O\big(dT(n)\big)$ time, where $T(n)$ is the running time of stableSort

Algorithm $\text{lexicographicSort}(S)$
Input: Sequence S of d-tuples
Output: Sequence S sorted in lexicographic order

1. **for** $i \leftarrow d \ldots 1$ **do**
2. $\quad\text{stableSort}(S, C_i)$

RADIX-SORT

- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O\big(d(n+N)\big)$



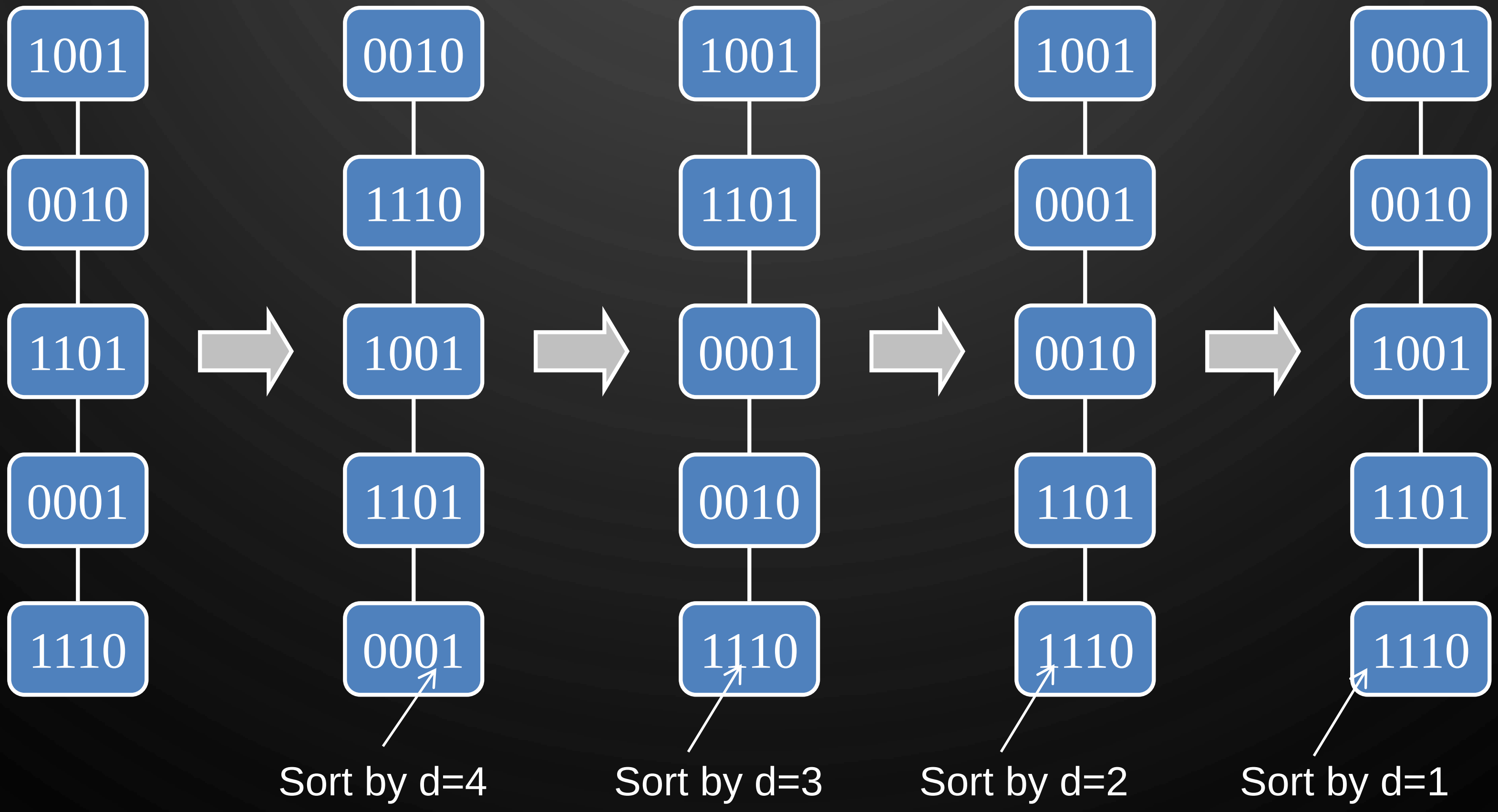
Algorithm $\underline{\text{radixSort}(S, N)}$
Input: Sequence S of d-tuples such that
$(0, \dots, 0) \leq (x_1, \dots, x_d)$ and
$(x_1, \dots, x_d) \leq (N-1, \dots, N-1)$
for each tuple $(x_1, \dots, x_d)$ in S
Output: Sequence S sorted in lexicographic order

1. **for** $i \leftarrow d \dots 1$ **do**
2. set the key k of each entry $\big(k, (x_1, \dots, x_d)\big)$ of S to ith dimension x_i
3. $\text{bucketSort}(S, N)$

EXAMPLE

RADIX-SORT FOR BINARY NUMBERS

- Sorting a sequence of 4-bit integers
  - $d = 4, N = 2$ so $O\big(d(n+N)\big) = O\big(4(n+2)\big) = O(n)$



SUMMARY OF SORTING ALGORITHMS

Algorithm	Time	Notes
Selection Sort	$O(n^2)$	Slow, in-place For small data sets
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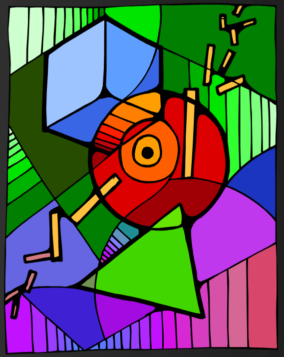
Heap Sort	$O(n \log n)$	Fast, in-place For large data sets
Quick Sort	Exp. $O(n \log n)$ AC, BC $O(n^2)$ WC	Fastest, randomized, in-place For large data sets
Merge Sort	$O(n \log n)$	Fast, sequential data access For huge data sets
Radix Sort	$O\left(d(n+N)\right)$, d #digits, N range of digit values	Fastest, stable only for integers

SETS

SET OPERATIONS



- A set is an ordered data structure similar to an ordered map, except only elements are stored (and yes elements must be unique)
- We represent a set by the sorted sequence of its elements
- By specializing the auxiliary methods the generic merge algorithm can be used to perform basic set operations:
    - Union - $A \cup B$ – Return all elements which appear in A or B (unique only)
    - Intersection - $A \cap B$ – Return only elements which appear in both A and B
    - Subtraction - $A \setminus B$ – Return elements in A which are not in B
- The running time of an operation on sets A and B should be at most $O(n_A + n_B)$

- Set union:
    - if $a < b$
      $$S.\text{insertFront}(a)$$
    - if $b < a$
      $$S.\text{insertFront}(b)$$
    - else $a = b$
      $$S.\text{insertFront}(a)$$
- Set intersection:
    - if $a < b$
      $$\{do\ nothing\}$$
    - if $b < a$
      $$\{do\ nothing\}$$
    - else $a = b$
      $$S.\text{insertBack}(a)$$

GENERIC MERGING

- Generalized merge of two sorted sets A and B
- Auxiliary methods (generic functions)
  - $\mathrm{aIsLess}(a, S)$
  - $\mathrm{bIsLess}(b, S)$
  - $\mathrm{bothAreEqual}(a, b, S)$
- Runs in $O(n_A + n_B)$ time provided the auxiliary methods run in $O(1)$ time

Algorithm $\mathrm{genericMerge}(A, B)$

Input: Sets A, B (implemented as sequences)

Output: Set S

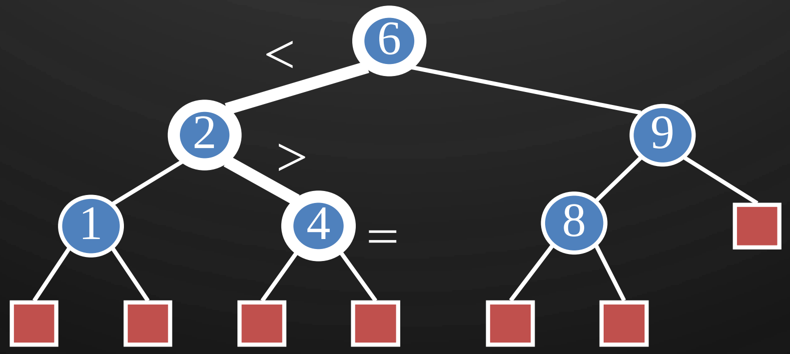
```
S ← ∅
while ¬A.empty( ) ∧ ¬B.empty( ) do
    a ← A.front( ); b ← B.front( )
    if a < b
        aIsLess(a, S) //generic action
        A.eraseFront( );
    else if b < a
        bIsLess(b, S) //generic action
        B.eraseFront( )
    else //a = b
        bothAreEqual(a, b, S) //generic action
        A.eraseFront( ); B.eraseFront( )
while ¬A.empty() do
    aIsLess(A.front( ), S); A.eraseFront( )
while ¬B.empty() do
    bIsLess(B.front( ), S); B.eraseFront( )
return S
```

USING GENERIC MERGE FOR SET OPERATIONS

- Any of the set operations can be implemented using a generic merge
- For example:
  - For intersection: only copy elements that are duplicated in both list
  - For union: copy every element from both lists except for the duplicates
- All methods run in linear time

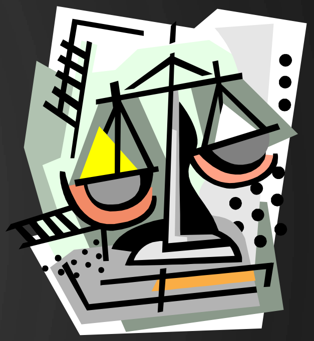
BETTER/TYPICAL SET IMPLEMENTATION

- Can use search trees such that the key is equivalent to the element to implement a set, allows fast ordering of data



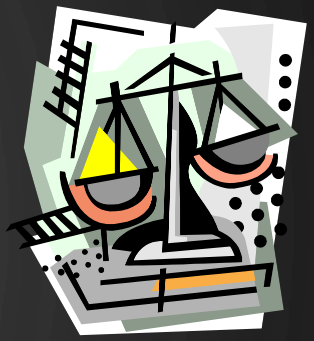
SELECTION

THE SELECTION PROBLEM



- Given an integer k and n elements $\{x_1, x_2, \ldots, x_n\}$, taken from a total order, find the k-th smallest element in this set.
  - Also called order statistics, the ith order statistic is the ith smallest element
  - Minimum - $k = 1$ - 1st order statistic
  - Maximum - $k = n$ - nth order statistic
  - Median - $k = \left\lfloor \frac{n}{2} \right\rfloor$
  - etc

THE SELECTION PROBLEM



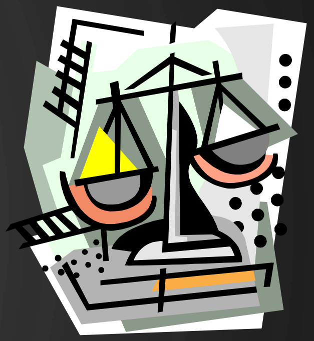
- Naïve solution - SORT!
- We can sort the set in $O(n \log n)$ time and then index the k-th element.

7 4 9 6 2 → 2 4 6 7 9

k=3

- Can we solve the selection problem faster?

THE MINIMUM (OR MAXIMUM)



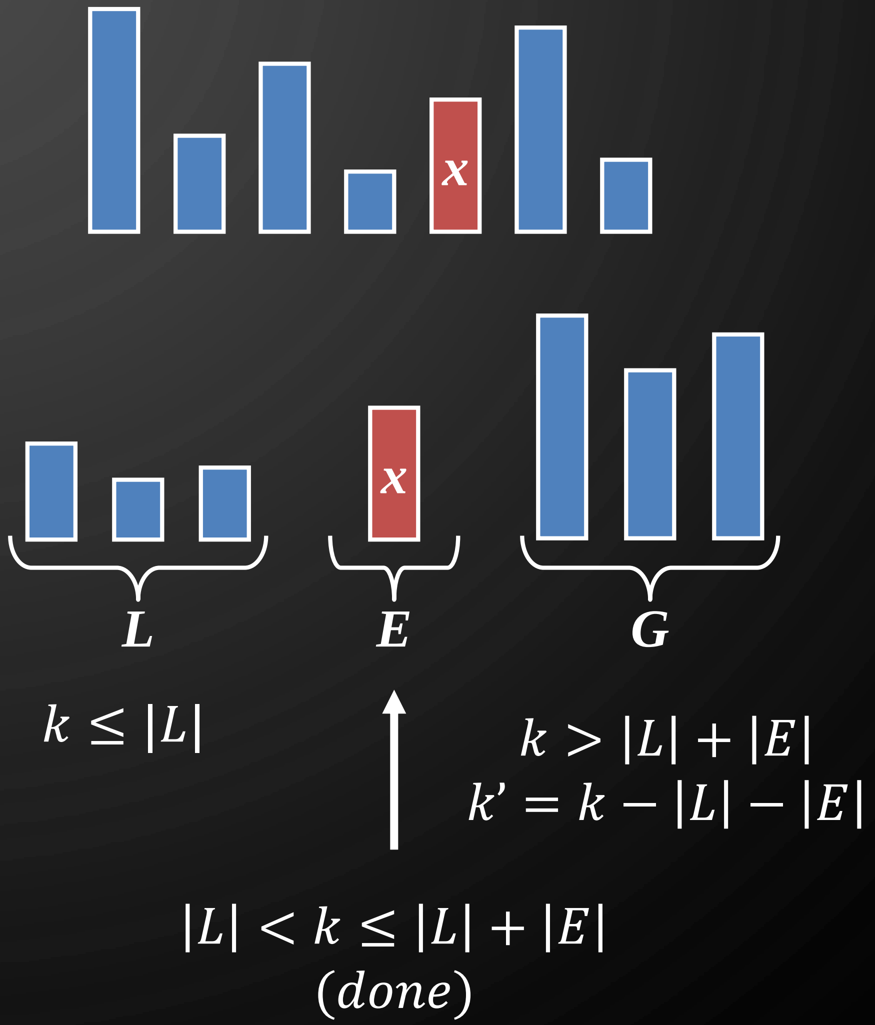
Algorithm minimum(A)
Input: Array A
Output: minimum element in A

```
m ← A[1]
for i ← 2 … n do
    m ← min(m, A[i])
return m
```

- Running Time
  - $O(n)$
- Is this the best possible?

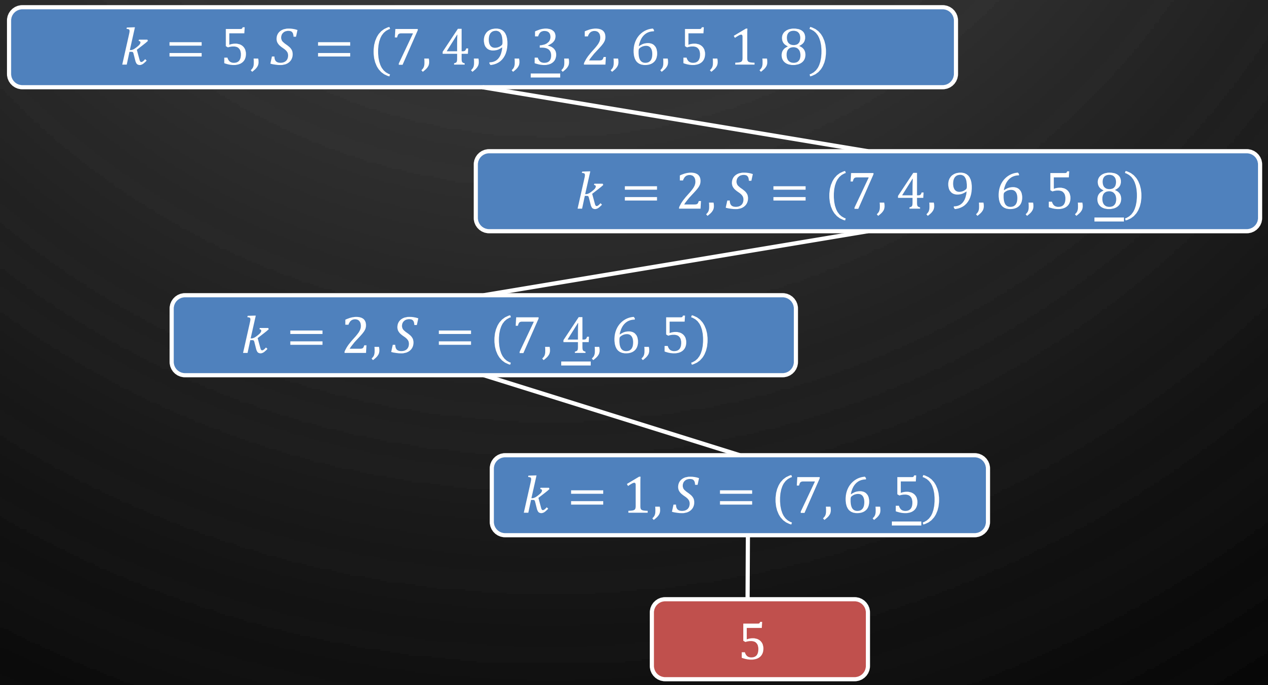
QUICK-SELECT

- Quick-select is a randomized selection algorithm based on the prune-and-search paradigm:
  - Prune: pick a random element x (called pivot) and partition S into
    - L elements $< x$
    - E elements $= x$
    - G elements $> x$
  - Search: depending on k, either answer is in E, or we need to recur on either L or G
- Note: Partition same as Quicksort

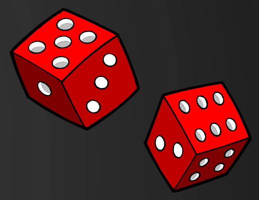


QUICK-SELECT VISUALIZATION

- An execution of quick-select can be visualized by a recursion path
  - Each node represents a recursive call of quick-select, and stores k and the remaining sequence

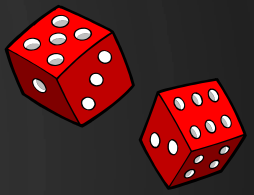


EXERCISE



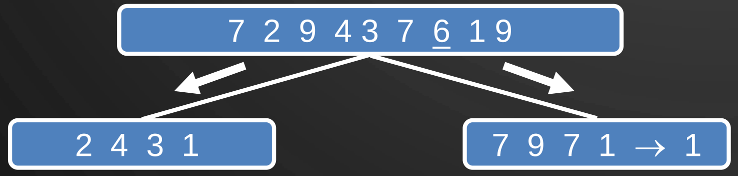
- Best Case - even splits (n/2 and n/2)
- Worst Case - bad splits (1 and n-1)

- Derive and solve the recurrence relation corresponding to the best case performance of randomized quick-select.
- Derive and solve the recurrence relation corresponding to the worst case performance of randomized quick-select.

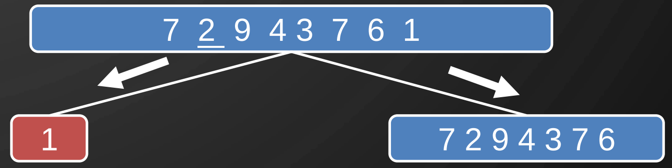


EXPECTED RUNNING TIME

- Consider a recursive call of quick-select on a sequence of size s
  - Good call: the size of L and G is at most $\frac{3s}{4}$
  - Bad call: the size of L and G is greater than $\frac{3s}{4}$



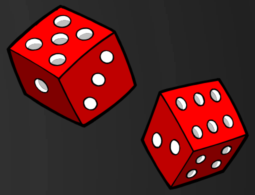
Good call



Bad call

- A call is good with probability $1/2$
  - $1/2$ of the possible pivots cause good calls:

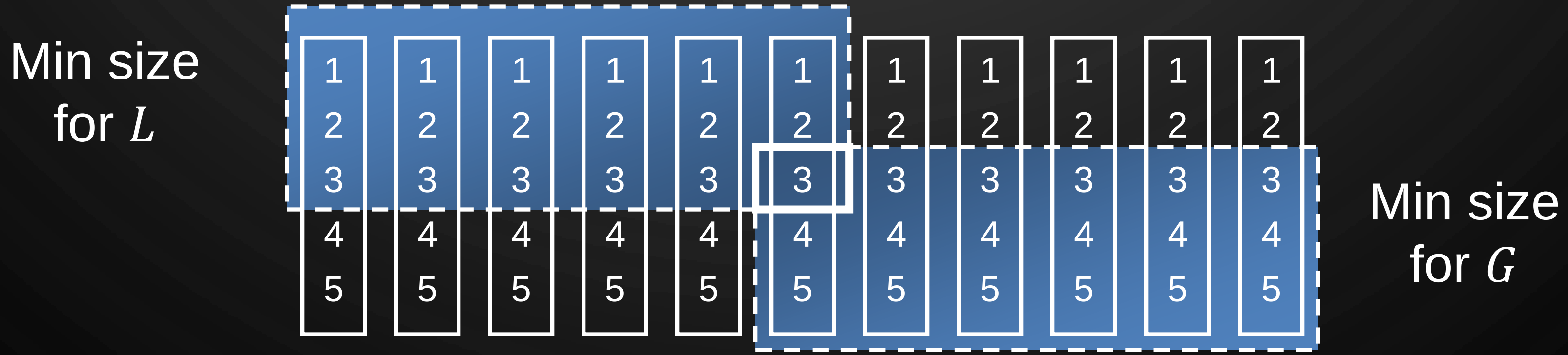
EXPECTED RUNNING TIME



- Probabilistic Fact #1: The expected number of coin tosses required in order to get one head is two
- Probabilistic Fact #2: Expectation is a linear function:
  - $E(X+Y) = E(X) + E(Y)$
  - $E(cX) = cE(X)$
- Let $T(n)$ denote the expected running time of quick-select.
- By Fact #2, $T(n) < T\left(\frac{3n}{4}\right) + bn * (expected\ \#\ of\ calls\ before\ a\ good\ call)$
- By Fact #1, $T(n) < T\left(\frac{3n}{4}\right) + 2bn$
- That is, $T(n)$ is a geometric series: $T(n) < 2bn + 2b\left(\frac{3}{4}\right)n + 2b\left(\frac{3}{4}\right)^2 n + 2b\left(\frac{3}{4}\right)^3 n + \cdots$
- So $T(n)$ is $O(n)$.
- We can solve the selection problem in $O(n)$ expected time.

DETERMINISTIC SELECTION

- We can do selection in $O(n)$ worst-case time.
- Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select:
  - Divide S into $\frac{n}{5}$ sets of 5 each
  - Find a median in each set
  - Recursively find the median of the “baby” medians.



- See Exercise C-11.22 for details of analysis.

INTERVIEW QUESTION 1

- You are given two sorted arrays, A and B, where A has a large enough buffer at the end to hold B. Write a method to merge B into A in sorted order.

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.

INTERVIEW QUESTION 2

- Write a method to sort an array of strings so that all the anagrams are next to each other.
  - Two words are anagrams if they use the exact same letters, i.e., race and care are anagrams

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.

INTERVIEW QUESTION 3

- Imagine you have a 2 TB file with one string per line. Explain how you would sort the file.

GAYLE LAAKMANN MCDOWELL, "CRACKING THE CODE INTERVIEW: 150 PROGRAMMING QUESTIONS AND SOLUTIONS", 5TH EDITION, CAREERCUP PUBLISHING, 2011.